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STATION KEEPING AT THE L4 LIBRATION POINT: A THREE-DIMENSIONAL STUDY

THESIS

AFIT/GA/AA/77D-3

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Preface

This paper describes the performance of a satellite controlled by a fixed gain feedback control system. Previous AFIT theses studying libration point motion and station keeping have provided an invaluable wealth of expertise as well as providing verification for some of the equations developed in this study.

I wish to express my thanks to Dr. Lynn Wolaver, my thesis advisor, for sharing his knowledge and for providing insight and inspiration when there seemed to be little light at the end of the tunnel. I also wish to thank Nino Baldachi for his personal encouragement and support.

George DeFilippi

<u>Contents</u>

Contents

List of Figures

List of Tables

List of Symbols

A_{ij}	element of the A matrix (ith row, jth column)
A, B	system equation matrices
C_1 - C_{42}	coefficients of the linearized equations
F_E, F_M, F_S	gravitational forces of earth, moon and sun referenced to the inertial frame
F_e, F_m, F_s	gravitational forces of earth, moon and sun referenced to the rotating frame
F, Q, R	matrices of the quadratic cost function
g_0	sea level gravity
I_{sp}	specific impulse
I, J, K	inertial coordinate system - sun centered
i, j, k	inertial coordinate system - barycenter centered
J	quadratic cost function
K, K_{ij}	gain matrix, gain matrix element
L_1, L_2 ...L_5	restricted three body libration points
M_e, M_m, M_s	mass of earth, moon, sun
m	mass of moon in nondimensionalized units
m_s	mass of the sun in nondimensionalized units
n	the constant, m_s/R_{sb}^3, in nondimensional units
N	mean angular rate, $\frac{2\pi}{T}$
P	potential function
q	drift weighting factor in quadratic cost function
R_{sb}	distance from sun to earth-moon barycenter
t	time
t_0	initial time; time of epoch
T, t_f	final time

$\underline{U}$	control vector
U_1, U_2, U_3	components of the control vector
U_{av}	average magnitude of the control vector
V	total velocity increment
w	radian frequency difference (ω-Ω)
x, y, z	rotating coordinate system centered at earth-moon barycenter; coordinates of satellite with respect to L4 point
x_o, y_o, 0	location of the L4 point with respect to earth-moon barycenter
$\underline{X}$	state vector
x_1, $x_2 \ldots x_9$	state variables
$\emptyset$	sun direction relative to rotating coordinate system
α	initial sun direction relative to rotating coordinate system
ω	angular velocity of moon about earth-moon barycenter
Ω	angular velocity of the earth-moon barycenter about the sun
ϵ, Ψ, ρ	coordinates of the vector from earth-moon barycenter to satellite
μ	eccentric anomaly of lunar orbit
e	eccentricity of lunar orbit
e_1	eccentricity of earth-moon barycenter orbit
μ_1	eccentric anomaly of earth-moon barycenter orbit

Abstract

In this work, the station keeping parameters at the earth-moon libration point, L4, were studied. First, the equations of motion for a three-dimensional, four body system with elliptical orbits were derived. These equations were then linearized about the L4 point; and optimal control theory was applied to obtain a linear feedback controller.

The major computations of the controller were associated with the gain matrix, which is the solution to the time varying Riccati equation. Because of the periodic nature of the time varying gains, it was felt that a modified (fixed gain) control could be used. The modified controller was found by computing the steady-state average of the time varying gains.

Several observations were made in studying the performance of the satellite in the vicinity of the L4 point. First, it was found that the modified controller was computationally much simpler than the optimum controller while providing near optimal performance. Second, there is approximately a linear relationship, up to a point, between stationkeeping cost and distance from the L4 point. Third, there are initial solar system configurations which minimize station keeping costs.

STATION KEEPING AT THE L4 LIBRATION POINT: A THREE-DIMENSIONAL STUDY

I. Introduction

The lunar libration points are equilibrium positions where a particle, if given the proper initial velocity, will remain fixed relative to the other two masses in a restricted three body system (see Figure 1). The collinear libration points, L_1, L_2, L_3, lie along the earth-moon line and have been shown to be unstable equilibrium points. The equilateral triangle points, L_4, L_5, have sides equal to the instantaneous earth-moon distance and are stable equilibrium points for mass ratios of less than approximately .04. While L_4 is the primary point studied in this work, the results could be easily applied to L_5 by symmetry.

In the solar system there are forces acting upon the satellite other than just the earth and moon. The libration points are no longer equilibrium positions. It is, however, convenient to think about the positions in space that correspond to the libration points. Stability and station keeping can be studied with reference to this point.

The lunar libration points, and L_4 in particular, have potential value as the exploration of space is expanded. In an Air War College report, Lt Col Berge (Ref 2:45-65) confirmed that the L_4 point has value as a communications link for interplanetary missions, back-side lunar missions, and a

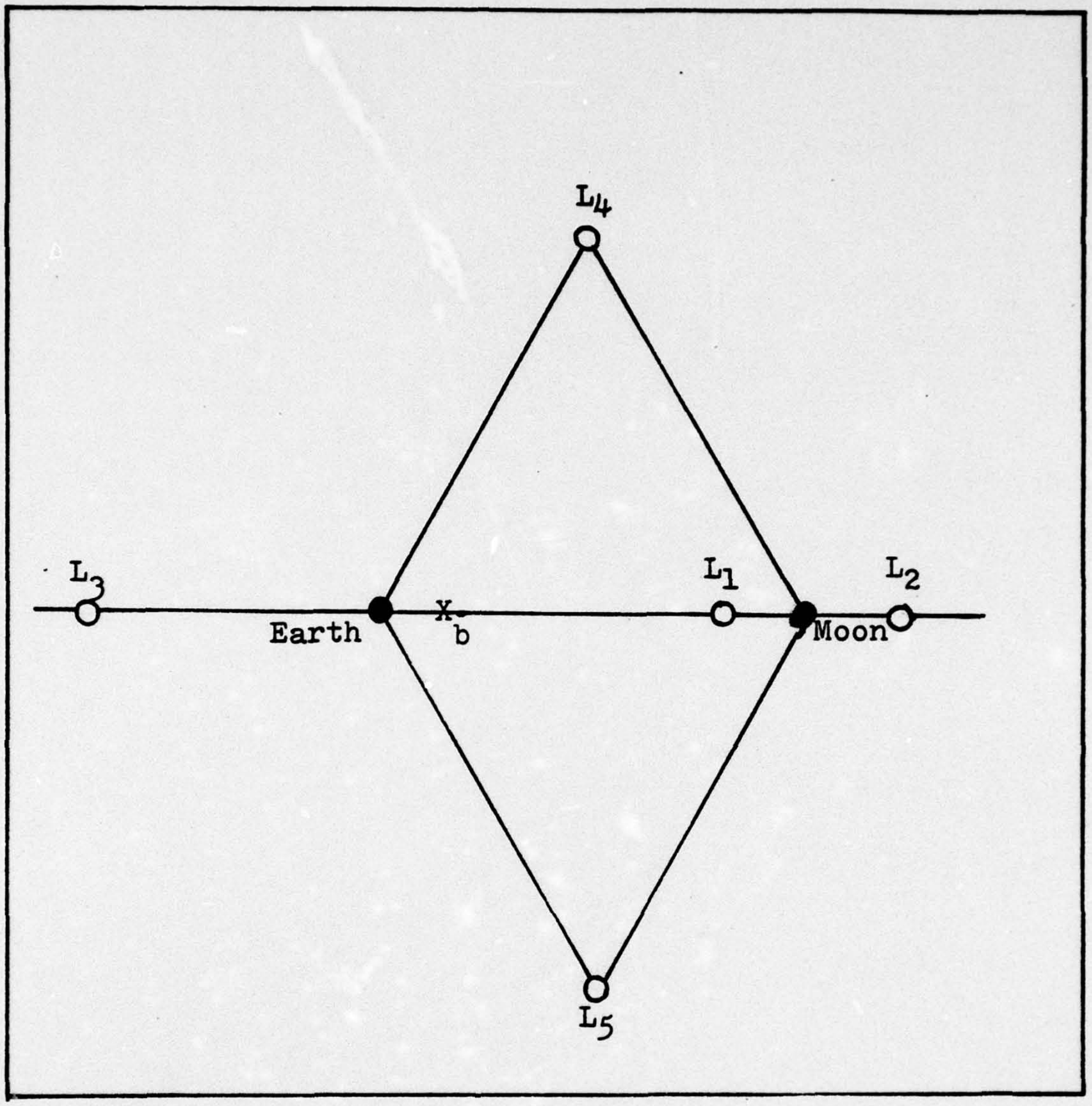


Figure 1

Restricted Three Body Libration Points

variety of scientific missions.

The object of this thesis is to continue the work of previous AFIT theses (Ref 6; 7; 8; 10). The equations of motion with respect to the L4 libration point will be derived based upon a three-dimensional four body system (earth-moon-sun-satellite) with elliptical orbits. And station keeping parameters will be examined relative to this model.

Formulation of the Equations of Motion

Considering the system of four bodies of Figure 2, the equations of motion referenced to the inertial, sun-centered coordinate system (I,J,K) can be written as

$$\Sigma F_I = M_p \ddot{\bar{r}}_{sp} = \bar{F}_E + \bar{F}_M + \bar{F}_S \tag{1}$$

where m_p is the mass of the satellite

$\bar{r}_{sp}$ is the vector from the sun to the satellite

$\bar{F}_E$, $\bar{F}_M$, $\bar{F}_S$ are the gravitational forces of the earth, moon and sun referenced to the inertial frame

The equations of motion can be written in terms of a rotating coordinate system (x,y,z) centered at the earth-moon barycenter (Ref Figure 2) by using the transformation matrix (Ref 13:432-435)

$$\underline{A} = \begin{vmatrix} \text{CoS } \beta \text{ CoS}(\theta + \phi) & \text{CoS } \beta \text{ Sin}(\theta + \phi) & \text{Sin } \beta \\ -\text{Sin } (\theta + \phi) & \text{CoS } (\theta + \phi) & 0 \\ -\text{Sin } \beta \text{ CoS}(\theta + \phi) & -\text{Sin } \beta \text{ Sin}(\theta + \phi) & \text{CoS } \beta \end{vmatrix}$$

where

β is the inclination of the earth-moon plane to the ecliptic

θ is the true anomaly of the earth-lunar orbit about the sun

ϕ is the sun direction relative to rotating frame

In terms of the rotating coordinate system, the equations of motion are

$$M_p \underline{A} \ddot{\bar{r}}_{sp} = \bar{F}_e + \bar{F}_m + \bar{F}_s \tag{2}$$

where the gravitational forces referenced to the rotating coordinate system are

$$\bar{F}_e = -\frac{k^2 M_e M_p}{r_{ep}^2}\bar{r}_{ep}; \quad \bar{F}_m = -\frac{k^2 M_m M_p}{r_{mp}^3}\bar{r}_{mp}; \quad \bar{F}_s = -\frac{k^2 M_s M_p}{r_{sp}^3}\bar{r}_{sp}$$

and (Ref Figure 3)

$$r_{ep}^2 = (r_e + \epsilon)^2 + \psi^2 + \rho^2; \; r_e = a_e\,(1 - e \cos \mu)$$

$$r_{mp}^2 = (r_m - \epsilon)^2 + \psi^2 + \rho^2; \; r_m = a_m\,(1 - e \cos \mu)$$

where

r_e is the earth - barycenter distance

r_m is the moon - barycenter distance

r_{ep} is the earth - satellite distance

r_{mp} is the moon - satellite distance

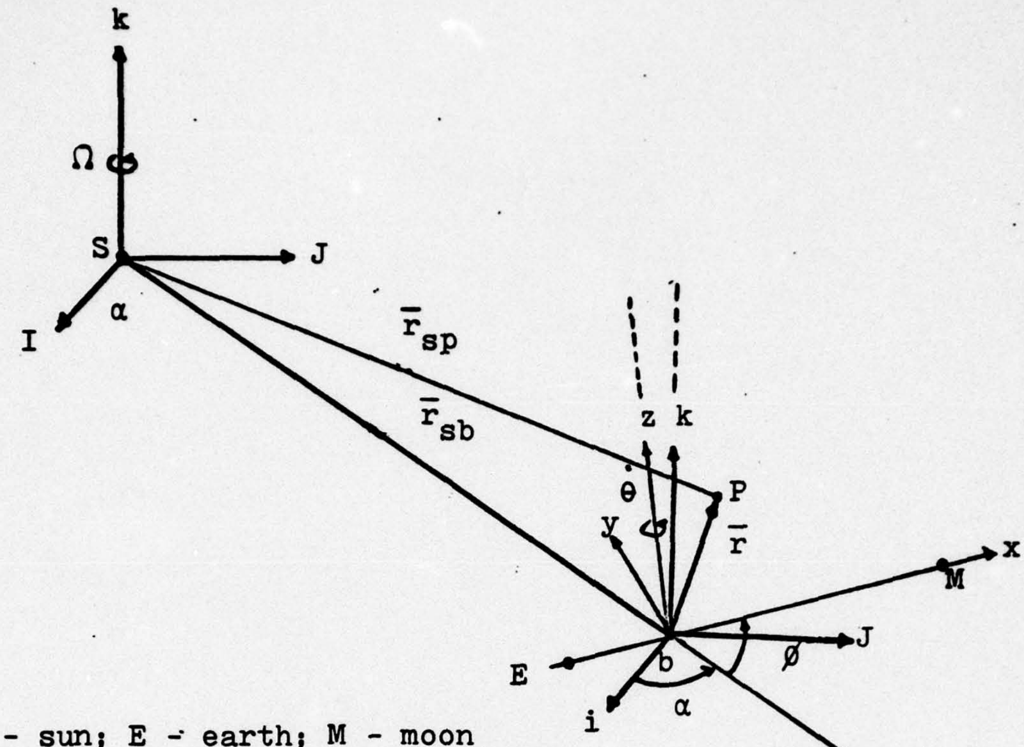
a_e is the semi-major axis of the earth orbit

a_m is the semi-major axis of the lunar orbit

e lunar eccentricity

μ - lunar eccentric anomaly

ϵ, ψ, ρ - are the components of the vector from earth-moon barycenter to satellite, $\bar{R}$, in the x,y,z-frame



S - sun; E - earth; M - moon

I,J,K - inertial axis, sun centered

i,j,k - inertial axis; barycenter centered

x,y,z - rotating axis; barycenter centered

α - initial sun direction relative to I,J,K-frame

Ω - angular velocity of rotating frame relative to inertial frame

$\dot{\theta}$ - angular velocity of rotating frame

ø - direction of sun relative to i,j,k - frame

Figure 2

Four Body System Representation

But, from Figure 2

$$\bar{r}_{sp} = \bar{r}_{sb} + \bar{r}$$

Differentiating

$$\ddot{\bar{R}}_{sb} = \underline{A}\,\ddot{\bar{r}}_{sb}$$

$$\ddot{\bar{r}} = (\bar{\omega}_x^2 + \dot{\bar{\omega}}_x)\bar{R} + 2\bar{\omega}_x\dot{\bar{R}} + \ddot{\bar{R}}$$

where the cross-product matrices $\bar{\omega}_x$ and $\dot{\bar{\omega}}_x$ are defined as (Ref 13:20)

$$\bar{\omega}_x = \begin{vmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \; ; \; \dot{\bar{\omega}}_x = \begin{vmatrix} 0 & -\ddot{\theta} & 0 \\ \ddot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

where

$\bar{r}_{sp}$ - vector from sun to satellite

$\bar{R}_{sb}$ - vector from sun to earth-moon barycenter referenced to rotating coordinate system (x,y,z)

$\bar{r}_{sb}$ - vector from sun to earth-moon barycenter, referenced to inertial coordinate system (I,J,K)

$\bar{R}$ - vector from earth-moon barycenter to satellite, referenced to x,y,z-frame

$\bar{r}$ - vector from earth-moon barycenter to satellite, referenced to I,J,K-frame

Substituting the above expressions into equation (2), the equations of motion can be written as

$$(\bar{\omega}_x^2 + \dot{\bar{\omega}}_x)\,\bar{R} + 2\bar{\omega}_x\dot{\bar{R}} + \ddot{\bar{R}} = -\frac{k^2M_e}{r_{ep}^3}\,\bar{r}_{ep} - \frac{k^2M_m}{r_{mp}^3}\,\bar{r}_{mp} + \frac{\bar{F}_s}{m_p} - \underline{A}\,\ddot{\bar{r}}_{sb} \qquad (3)$$

Defining the following quantities

$$\delta\overline{F}_s = \frac{\overline{F}_s}{M_p} - \underline{\underline{A}}\,\ddot{\overline{r}}_{sb}$$

$$V = \frac{M_e}{r_{ep}} + \frac{M_m}{r_{mp}}$$

and (Ref Figure 3)

$$\overline{R} = \begin{vmatrix} \epsilon \\ \psi \\ \rho \end{vmatrix} \;;$$

where

$\delta\overline{F}_s$ Solar perturbative force on satellite

V Gravitational potential function

M_e, M_m - Mass of earth, moon

equation (3) can be expressed as

$$\begin{aligned} \ddot{\epsilon} - 2\dot{\theta}\dot{\psi} - \ddot{\theta}\psi &= \frac{\partial V}{\partial \epsilon} + \dot{\theta}^2\epsilon + \delta F_{s\epsilon} \\ \ddot{\psi} + 2\dot{\theta}\dot{\epsilon} + \ddot{\theta}\epsilon &= \frac{\partial V}{\partial \psi} + \dot{\theta}^2\psi + \delta F_{s\psi} \\ \ddot{\rho} &= \frac{\partial V}{\partial \rho} + \delta F_{s\rho} \end{aligned} \tag{4}$$

For convenience of notation, a potential function can be written as (Ref 12:9)

$$P = \frac{M_e}{r_{ep}} + \frac{M_m}{r_{mp}} + \tfrac{1}{2}\,\dot{\theta}^2\,(\epsilon^2 + \psi^2)$$

Using this notation the equations of motion are

$$\begin{aligned} \ddot{\epsilon} - 2\dot{\theta}\dot{\psi} - \ddot{\theta}\psi &= \frac{\partial P}{\partial \epsilon} + \delta F_{s\epsilon} \\ \ddot{\psi} + 2\dot{\theta}\epsilon + \ddot{\theta}\epsilon &= \frac{\partial P}{\partial \psi} + \delta F_{s\psi} \\ \overline{\ddot{\rho}} &= \frac{\partial P}{\partial \rho} + \delta\overline{F}_{s\rho} \end{aligned} \tag{5}$$

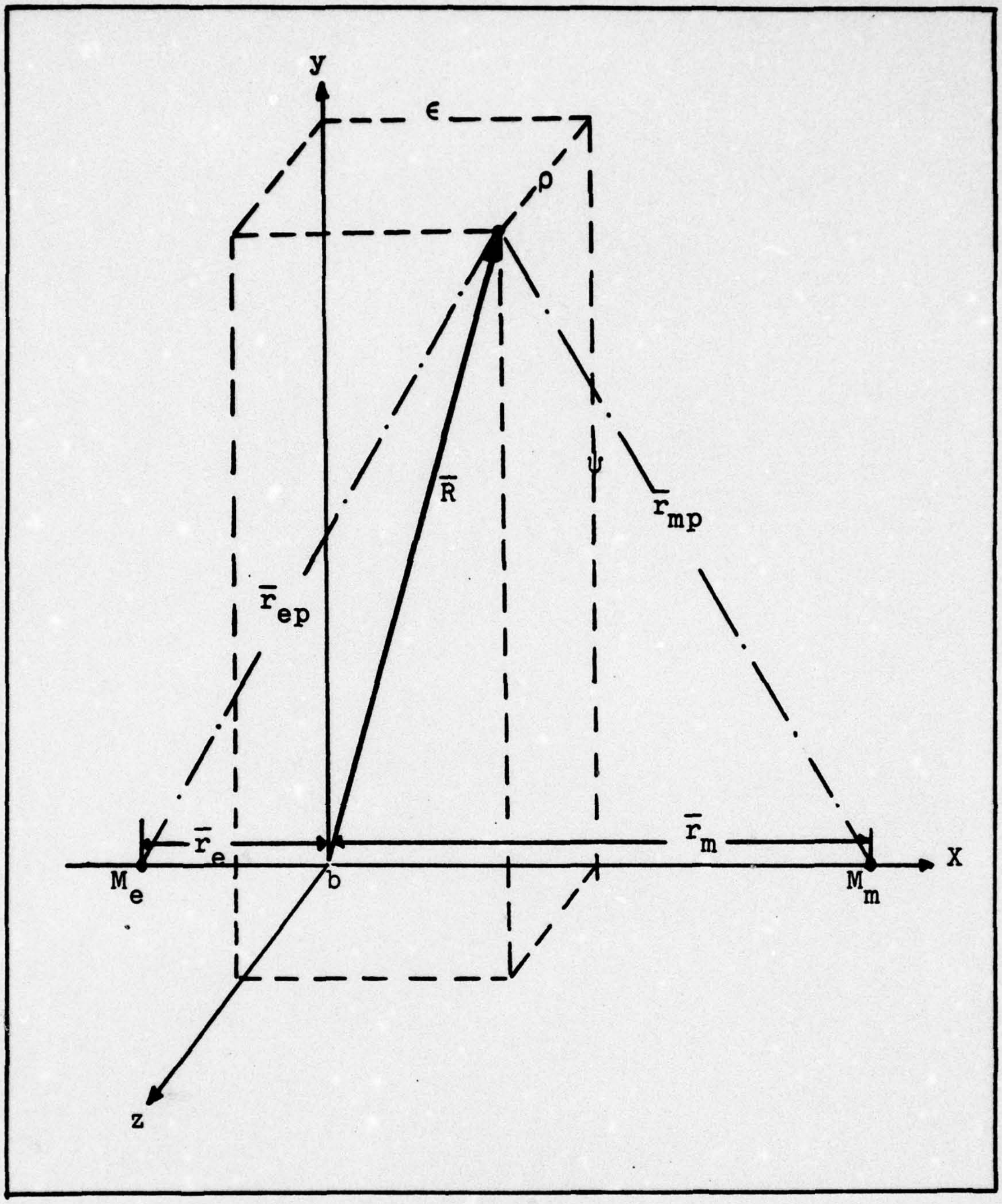


Figure 3

Satellite Position in Rotating

Coordinate System

Equations (5) are the equations of motion of the satellite with respect to the earth-moon barycenter. These equations include the following assumptions. First, the center of mass of the sun is the center of mass of the four body solar system. Second, the earth and moon are assumed to move in perfect elliptical orbits about their barycenter; and the earth-moon barycenter moves in an elliptical orbit about the sun. Third, all masses are assumed to be point masses.

One can also note that equations (5) represent the superposition of the differential solar force, $\delta\bar{F}_s$, and the three body equations of motion. Thus, the derivation can be completed in two steps. First, the three body equations can be linearized about the L_4 point. Next, the differential solar force can be linearized about L_4. Once these expressions are obtained, they can be substituted into equation (5) to give the linearized four body equations with respect to the L_4 point.

<u>Three Body Equations of Motion</u>

The three body equations can be obtained from equation (5) by letting the differential solar force, $\delta\bar{F}_s$, be zero (Ref 12:1). This gives

$$
\begin{aligned}
\ddot{\epsilon} - 2\dot{\theta}\psi - \ddot{\theta}\psi &= \frac{\partial P}{\partial \epsilon} \\
\ddot{\psi} + 2\dot{\theta}\dot{\epsilon} + \ddot{\theta}\epsilon &= \frac{\partial P}{\partial \psi} \qquad (6) \\
\ddot{\rho} &= \frac{\partial P}{\partial \rho}
\end{aligned}
$$

The units can be non-dimensionalized by choosing the mass unit to be the total mass of the earth and moon ($M_e + M_m = 1$) and the distance unit to be the earth-moon distance at closest approach. Taking epoch ($t_o = 0$) to be the time of closest approach, Kepler's law gives

$$\frac{k^2(M_e + M_M)}{a^3} = N^2$$

where

N - mean angular rate, 2π/Period

a - semi-major axis of satellite orbit

k^2 - gravitational constant

Choosing the time scale such that k^2 and the lunar period are unity results in a time unit equal to 4.384 mean solar days; thus $N^2 = \frac{1}{a^3}$. But from Fig. 4, D = 1 = a (1-e) and thus $N^2 = (1 - e)^3$. The non-dimensional equations can then be written as

$$\ddot{\epsilon} - 2\dot{\theta}\dot{\psi} - \ddot{\theta}\psi = \frac{1}{N^2}\frac{\partial P}{\partial \epsilon}$$

$$\ddot{\psi} + 2\dot{\theta}\dot{\epsilon} + \ddot{\theta}\epsilon = \frac{1}{N^2}\frac{\partial P}{\partial \psi}$$

$$\ddot{\rho} = \frac{1}{N^2}\frac{\partial P}{\partial \rho}$$

where

θ is the true anomaly of the lunar orbit

$\dot{\theta}$ is the speed of the rotating frame relative to the inertial frame

Using Kepler's law, the angular velocity of the rotating coordinate system, $\dot{\theta}$, can be reduced to (Ref 12:3-4)

$$\dot{\theta} = \frac{\sqrt{1-e^2}}{(1-e \text{ Cos } \mu)^2}$$

Differentiating gives (Ref 12:3-4)

$$\ddot{\theta} = \frac{-2e \operatorname{Sin}\mu \sqrt{1-e^2}}{(1 - e \operatorname{Cos} \mu)^4}$$

when

e is the lunar eccentricity

μ is the lunar eccentric anomaly

For small perturbations (x,y,z) about the L_4 point $(x_o, y_o, 0)$, the coordinates of the satellite can be expressed as $x_o + x$, $y_o + y$, z (see Figure 4). The right hand side of equation (7) can be expanded about the equilibrium point $(x_o, y_o, 0)$ in a Taylor series as

$$\begin{aligned}
\frac{1}{N^2}\frac{\partial P}{\partial \epsilon} &= \frac{1}{N^2}\frac{\partial P}{\partial \epsilon}\bigg|_{\underline{x}_o} + \frac{1}{N^2}\left[\frac{\partial^2 P}{\partial \epsilon^2}\bigg|_{\underline{x}_o} X + \frac{\partial^2 P}{\partial \epsilon\,\partial \psi}\bigg|_{\underline{x}_o} Y + \frac{\partial^2 P}{\partial \epsilon\,\partial \rho}\bigg|_{\underline{x}_o} z + \ldots\right] \\
\frac{1}{N^2}\frac{\partial P}{\partial \psi} &= \frac{1}{N^2}\frac{\partial P}{\partial \psi}\bigg|_{\underline{x}_o} + \frac{1}{N^2}\left[\frac{\partial^2 P}{\partial \psi\,\partial \epsilon}\bigg|_{\underline{x}_o} X + \frac{\partial^2 P}{\partial \psi^2}\bigg|_{\underline{x}_o} Y + \frac{\partial P}{\partial \psi\,\partial \rho}\bigg|_{\underline{x}_o} z + \ldots\right] \qquad (8) \\
\frac{1}{N^2}\frac{\partial P}{\partial \rho} &= \frac{1}{N^2}\frac{\partial P}{\partial \rho}\bigg|_{\underline{x}_o} + \frac{1}{N^2}\left[\frac{\partial^2 P}{\partial \rho\,\partial \epsilon}\bigg|_{\underline{x}_o} X + \frac{\partial^2 P}{\partial \rho\,\partial \psi}\bigg|_{\underline{x}_o} Y + \frac{\partial^2 P}{\partial \rho^2}\bigg|_{x_o} z + \ldots\right].
\end{aligned}$$

where

$\epsilon = x_o + x$

$\psi = y_o + y$

$\rho = z$

$\underline{x}_o$ = short hand notation for coordinates of the L4 point (x_o, y_o, o)

It is assumed that the perturbations (x,y,z) from the L_4 point will be small compared to the distance from L_4 (x_o, y_o, o) to the earth-moon barycenter. Also, the terms x^2, y^2, z^2, xy, xz, yz, will be much smaller than x, y, or z

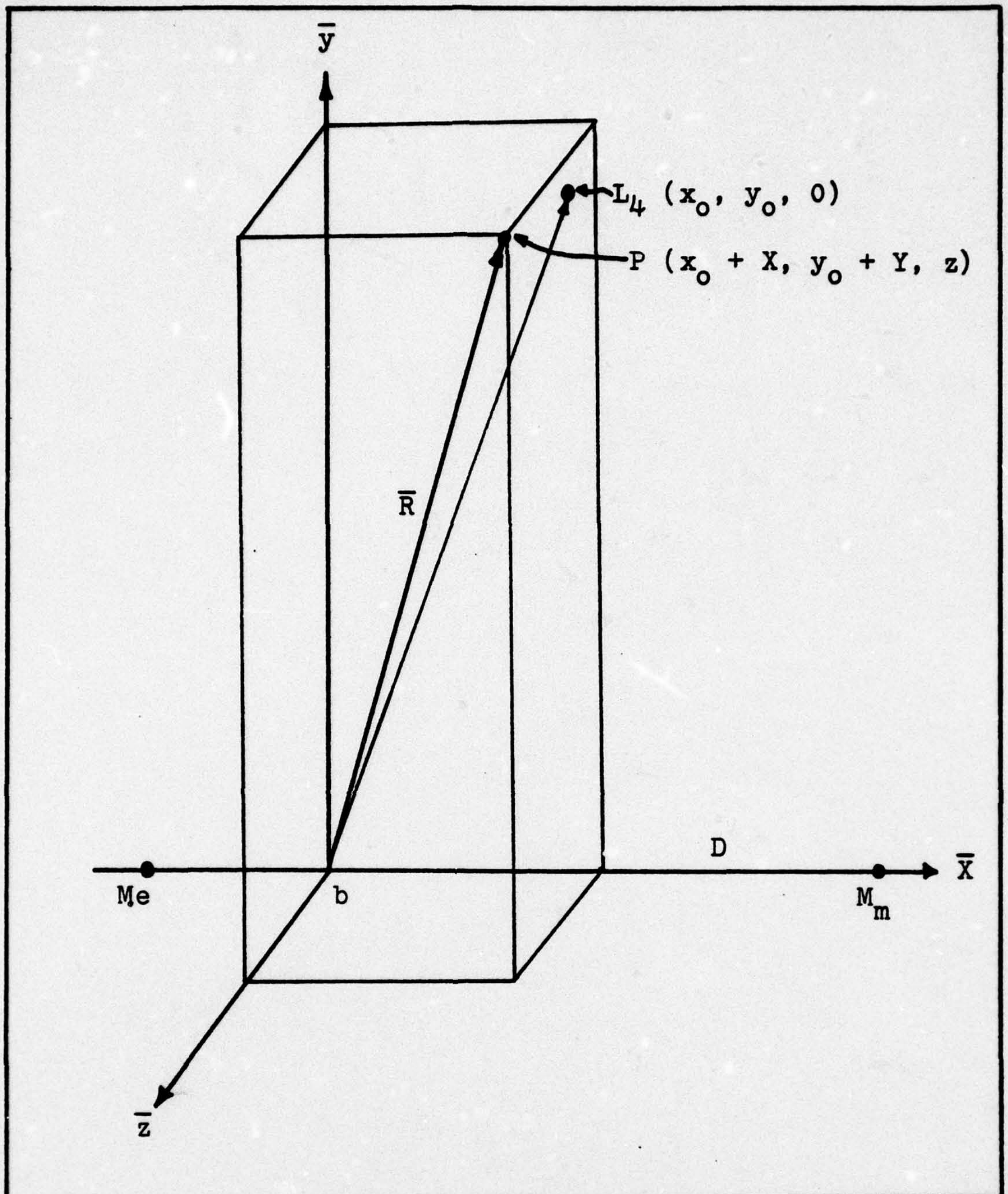


Figure 4

Satellite Position Relative to L4

and can be neglected. Only the zero and first order terms of equation (8) will be retained.

Making use of the following relationships (Ref 12:4-8)

$$r_m = \frac{1 - e \cos \mu}{(1 - e)}; \quad r_e = \frac{m\,(1 - e \cos \mu)}{(1 - e)}$$

it can be shown (Ref 12:6-8) that at the equilibrium point $(x_o, y_o, 0)$ the distances are:

$$\epsilon = x_o = \frac{(1 - 2m)\,(1 - e \cos \mu)}{2\,(1 - e)}; \quad \psi = y_o = \frac{\sqrt{3}\,(1 - e \cos \mu)}{2\,(1-e)}$$

$$N^2 = (1 - e)^3$$

where

μ is the eccentric anomaly of the lunar orbit

e is the eccentricity of the lunar orbit

m is the mass of the moon (non-dimensionalized)

the partial derivatives of equation (8) can be evaluated at the equilibrium point $(x_o, y_o, 0)$ as (Ref 12:7 - 8)

$$\frac{1}{N^2} \frac{\partial^2 P}{\partial \epsilon^2}\bigg|_{\underline{x}_o} = \frac{1}{4(1 - e \cos \mu)^3} + \frac{(1 - e^2)}{(1 - e \cos \mu)^4}$$

$$\frac{1}{N^2} \frac{\partial^2 P}{\partial \psi^2}\bigg|_{\underline{x}_o} = \frac{5}{4(1 - e \cos \mu)^3} + \frac{(1 - e^2)}{(1 - e \cos \mu)^4}$$

$$\frac{1}{N^2} \frac{\partial^2 P}{\partial \rho^2}\bigg|_{\underline{x}_o} = - \frac{1}{(1 - e \cos \mu)^3}$$

$$\frac{1}{N^2} \frac{\partial^2 P}{\partial \epsilon\, \partial \psi}\bigg|_{\underline{x}_o} = \frac{1}{N^2} \frac{\partial^2 P}{\partial \psi\, \partial \epsilon}\bigg|_{\underline{x}_o} = \frac{3\sqrt{3}}{4} \frac{(1 - 2m)}{(1 - e \cos \mu)^3}$$

$$\frac{1}{N^2} \frac{\partial^2 P}{\partial \epsilon\, \partial \rho}\bigg|_{\underline{x}_o} = \frac{1}{N^2} \frac{\partial^2 P}{\partial \rho\, \partial \epsilon}\bigg|_{\underline{x}_o} = \frac{1}{N^2} \frac{\partial^2 P}{\partial \psi\, \partial \rho}\bigg|_{\underline{x}_o} = \frac{1}{N^2} \frac{\partial^2 P}{\partial \rho\, \partial \psi}\bigg|_{\underline{x}_o} = 0$$

where

$$\underline{x}_o = \begin{vmatrix} x_o \\ y_o \\ 0 \end{vmatrix}$$

After evaluating the partial derivatives and noting that $(x_o, y_o, 0)$ is a solution to equation (7), the linearized equations of motion about the L_4 point are

$$\ddot{x} - [\,2\sqrt{1-e^2}\;F^2]\,\dot{y} + [2e\,\mathrm{Sin}\,\mu\sqrt{1-e^2}\;F^4]\,y = [\,(1-e^2)F^4 - \tfrac{1}{4}F^3]\,x + [\tfrac{3\sqrt{3}}{4}\,QF^3]\,y$$

$$\ddot{y} + [\,2\sqrt{1-e^2}\;F^2]\,\dot{x} - [2e\,\mathrm{Sin}\,\mu\sqrt{1-e^2}\;F^4]\,x = [\tfrac{3\sqrt{3}}{4}\,QF^3]\,x + [(1-e^2)F^4 + \tfrac{5}{4}F^3]\,y \qquad (9)$$

$$\ddot{z} = -F^3 z$$

where

$$Q = 1-2m;\; M_e + M_m = 1;\; m = \frac{M_m}{1}\;;\; F = 1/(1-e\,\mathrm{Cos}\,\mu)$$

<u>Restricted Three Body Equations of Motion</u>. It should be noted that the linearized equations of motion for the restricted three body problem can be easily obtained from equation (9) by letting the eccentricity, e, approach zero. The linearized restricted three body equations are (Ref 5: 22-23)

$$\ddot{x} - 2\dot{y} = \tfrac{3}{4}\,x + \tfrac{3\sqrt{3}}{4}\,(1 - 2m)y$$

$$\ddot{y} + 2\dot{x} = \tfrac{3\sqrt{3}}{4}\,x + \tfrac{9}{4}\,y$$

$$\ddot{z} = -z$$

Four Body Equations of Motion

To complete the derivation of the four body equations of motion linearized about the L_4 point, an expression for the differential solar force, $\delta\bar{F}_s$, linearized about the L_4 point is needed. $\delta\bar{F}_s$ represents the difference between the solar gravitational attraction of the sun on the satellite and the solar gravitational attraction on the earth-moon barycenter. The derivation and linearization are presented in detail in Appendix A. The linearized expression for δF_s is summarized here as

$$\delta\bar{F}_s = -N \begin{vmatrix} x_o + x \\ y_o + y \\ z \end{vmatrix} + 3N \begin{vmatrix} \frac{(x_o + x)}{4} \quad (1 + \text{Cos } 2\beta) - \frac{z}{4} \text{ Sin } 2\beta \\ \frac{y_o + y}{2} \\ \frac{-(x_o + x)}{4} \quad \text{Sin } 2\beta + \frac{z}{4} (1 - \text{Cos } 2\beta \end{vmatrix} \cdot$$

$$(1 + \text{Cos } 2\emptyset) +$$

$$+ \begin{vmatrix} \frac{-(y_o + y)}{2} \quad \text{Cos } \beta \\ \frac{-(x_o + x)}{2} \quad \text{Cos } \beta + \frac{z}{4} \text{ Sin } \beta \\ \frac{(y_o + y)}{2} \quad \text{Sin } \beta \end{vmatrix} (3N \text{ Sin } 2\emptyset)$$

$$+ \frac{N x_o y_o}{R_{sb}} \begin{vmatrix} \frac{3}{4} S\emptyset + \frac{15}{4} \text{ Cos } 2\beta \text{ Sin } \emptyset \\ -\frac{9}{2} \text{ Cos } \beta \text{ Cos } \emptyset \\ -\frac{15}{4} \text{ Sin } 2\beta \text{ Sin } \emptyset \end{vmatrix}$$

$$
+ \frac{9N}{8 R_{sb}} \begin{vmatrix} \frac{1}{2} x_o^2 (\text{Cos } \beta - 5 \text{ Cos } 3\beta) - 2y_o^2 \text{ Cos } \beta \quad \text{Cos } \phi \\ \frac{1}{3} x_o^2 (1 + 5 \text{ Cos } 2\beta) + y_o^2 \quad \text{Sin } \phi \\ (\frac{7}{6} x_o^2 + 2y_o^2) \text{ Sin } \beta \text{ Cos } \phi \end{vmatrix}
$$

$$
+ \frac{9Nx}{8 R_{sb}} \begin{vmatrix} \frac{1}{2} x_o (\text{Cos } \beta - 5 \text{ Cos } 3\beta) \text{ Cos } \phi + \frac{2}{3} y_o (1 + 5 \text{ Cos } 2\beta) \text{ Sin } \phi \\ \frac{2}{3} x_o (1 + 5 \text{ Cos } 2\beta) \text{ Sin } \phi - 4 y_o \text{ Cos } \beta \text{ Cos } \phi \\ \frac{7}{3} x_o \text{ Sin } \beta \text{ Cos } \phi - \frac{10}{3} y_o \text{ Sin } 2\beta \text{ Sin } \phi \end{vmatrix} \qquad (10)
$$

$$
+ \frac{9Ny}{8 R_{sb}} \begin{vmatrix} \frac{2}{3} x_o (1 + 5 \text{ Cos } 2\beta) \text{ Sin } \phi - 4 y_o \text{ Cos } \beta \text{ Cos } \phi \\ -4 x_o \text{ Cos } \beta \text{ Cos } \phi + 2 y_o \text{ Sin } \phi \\ - \frac{10}{3} x_o \text{ Sin } 2\beta \text{ Sin } \phi + 4 y_o \text{ Sin } \beta \text{ Cos } \phi \end{vmatrix}
$$

$$
+ \frac{9 Nz}{8} \begin{vmatrix} \frac{7}{3} x_o \text{ Sin } \beta \text{ Cos } \phi - \frac{10}{3} y_o \text{ Sin } 2\beta \text{ Sin } \phi \\ - \frac{10}{3} x_o \text{ Sin } 2\beta \text{ Sin } \phi + 4 y_o \text{ Sin } \beta \text{ Cos } \phi \\ - \frac{7}{3} x_o \text{ Cos } \beta \text{ Cos } \phi + \frac{2}{3} y_o \text{ Sin } \phi \end{vmatrix}
$$

where

$R_{sb} = a_s (1 - e_1 \text{ Cos } \mu_1)/(a_m (1 - e))$

$N = M_s/R_{sb}^3$

$x_o = (1 - 2m) (1 - e \text{ Cos } \mu)/ (1 - e) 2$

$y_o = (1 - e \text{ Cos } \mu)/(2 (1 - e))$

e - lunar orbit eccentricity

e_1 - earth-moon barycenter orbit eccentricity

μ - lunar orbit eccentric anomaly

μ_1 - earth-moon barycenter orbit eccentric anomaly

The nondimensionalized four body equations of motion with respect to the L4 point are obobtained by combining equations (9) and (10) as specified by equation (5). This results in

$$\ddot{x} - C_1 \dot{y} - (C_2 + C_{10} \text{ Cos } 2\phi + C_{12} \text{ Cos } \phi + C_{13} \text{ Sin } \phi)_x - (C_3 - C_{11} \text{ Sin } 2\phi - C_{15} \text{ Cos } \phi + C_{14} \text{ Sin } \phi) y - (-C_4 - C_4 \text{ Cos } 2\phi + C_{16} \text{ Cos } \phi - C_{17} \text{ Sin } \phi)_z = C_5 + C_6 \text{ Cos } 2\phi - C_7 \text{ Sin } 2\phi + C_8 \text{ Cos } \phi + C_9 \text{ Sin } \phi \quad (11)$$

$$\ddot{y} + C_1 \dot{x} - (C_{10} - C_{11} \text{ Sin } 2\phi - C_{26} \text{ Cos } \phi + C_{14} \text{ Sin } \phi)_x - (C_{19} - C_{25} \text{ Cos } 2\phi - C_{27} \text{ Cos } \phi + C_{28} \text{ Sin } \phi)_y - (-C_{29} \text{ Sin } \phi + C_{30} \text{ Cos } \phi + C_{31} \text{ Sin } 2\phi)_z = C_{20} - C_{21} \text{ Cos } 2\phi - C_{22} \text{ Sin } 2\phi - C_{23} \text{ Cos } \phi + C_{24} \text{ Sin } \phi \quad (12)$$

$$\ddot{z} - (-C_4 - C_4 \text{ Cos } 2\phi + C_{16} \text{ Cos } \phi - C_{40} \text{ Sin } \phi)_x - (C_{35} \text{ Sin } 2\phi - C_{29} \text{ Sin } \phi + C_{30} \text{ Cos } \phi)_y - (C_{32} + C_{36} \text{ Cos } 2\phi - C_{41} \text{ Cos } \phi + C_{42} \text{ Sin } \phi)_z = -C_{33} - C_{33} \text{ Cos } 2\phi + C_{34} \text{ Sin } 2\phi + C_{37} \text{ Cos } \phi - C_{38} \text{ Sin } \phi \quad (13)$$

where (in nondimensionalized units)

$Q = (1 - 2m)$ $\quad x_o = \frac{Q\,(1 - e \text{ Cos } \mu)}{2\,(1-e)}$ $\quad y_o = \frac{\sqrt{3}\,(1 - e \text{ Cos } \mu)}{2\,(1 - e)}$ $\quad N = \frac{M_s}{R_{sb}^3}$

$F = 1/(1 - e \text{ Cos } \mu)$ $\quad R_{sb} = a_s\,(1 - e_1 \text{ Cos } \mu_1)/(a_m\,(1-e))$

and $C_1 - C_{42}$ are listed in Table I.

Table I

Coefficients of the Equations of Motion

Term	Expression
C_1	$2\sqrt{1-e^2}\ F^2$
C_2	$(1-e^2)F^4 - \frac{1}{4}F^3 + \frac{N}{4}(3 \cos 2\beta - 1)$
C_3	$\frac{3\sqrt{3}}{4}QF^3 - 2e \sin \mu \sqrt{1-e^2}\ F^4$
C_4	$\frac{3N}{4} \sin 2\beta$
C_5	$\frac{N}{4} x_o (3 \cos 2\beta - 1)$
C_6	$\frac{3}{4} N x_o (1 + \cos 2\beta)$
C_7	$\frac{3}{2} n y_o \cos \beta$
C_8	$\frac{9n}{8 R_{sb}} \{\frac{1}{4} x_o^2 (C\beta - 5C3\beta) - 2y_o^2 C\beta\}$
C_9	$\frac{n}{R_{sb}} x_o y_o (\frac{3}{4} + \frac{15}{4} \cos 2\beta)$
C_{10}	$\frac{3}{4} N (1 + \cos 2\beta)$
C_{11}	$\frac{3}{2} n \cos \beta$
C_{12}	$\frac{9}{16} \frac{n}{R_{sb}} \{x_o (\cos \beta - 5 \cos 3\beta)\}$
C_{13}	$\frac{3}{4} \frac{n}{R_{sb}} y_o (1 + 5 \cos 2\beta)$
C_{14}	$\frac{3}{4} \frac{n}{R_{sb}} x_o (1 + 5 \cos 2\beta)$
C_{15}	$\frac{9}{2} \frac{n}{R_{sb}} y_o \cos \beta$

Table I - continued

Term	Expression
C_{16}	$\frac{21\ n}{8\ R_{sb}}\ x_o$ Sin β
C_{17}	$\frac{15}{4}\ \frac{n}{R_{sb}}\ y_o$ Sin 2β
C_{18}	$\frac{3\sqrt{3}}{4}\ QF^3 + 2e \sin\mu\ \sqrt{1-e^2}\ F^4$
C_{19}	$(1-e^2)F^4 + \frac{5}{4}\ F^3 + \frac{n}{2}$
C_{20}	$\frac{N}{2}\ y_o$
C_{21}	$\frac{3n}{2}\ y_o$
C_{22}	$\frac{3n}{2}\ x_o \cos\beta$
C_{23}	$\frac{9n}{2\ R_{sb}}\ x_o\ y_o \cos\beta$
C_{24}	$\frac{9n}{8\ R_{sb}}\ (\frac{1}{3}\ x_o^2\ (1 + 5e\ 2\beta) + y_o^2)$
C_{25}	$\frac{3n}{2}$
C_{26}	$\frac{9n}{2\ R_{sb}}\ y_o \cos\beta$
C_{27}	$\frac{9n}{2\ R_{sb}}\ x_o \cos\beta$
C_{28}	$\frac{9n}{4\ R_{sb}}\ y_o$
C_{29}	$\frac{15n}{4\ R_{sb}}\ x_o$ Sin 2β
C_{30}	$\frac{9n}{2\ R_{sb}}\ y_o$ Sin 2β
C_{31}	$\frac{3n}{4}$ Sin β

Table I - continued

Term	Expression
C_{32}	$-F^3 \ -\frac{n}{4}\ (1 + 3 \cos 2\beta)$
C_{33}	$\frac{3n}{4}\ x_o \ \mathrm{Sin}\ 2\beta$
C_{34}	$\frac{3n}{2}\ y_o \ \mathrm{Sin}\ 2\beta$
C_{35}	$\frac{3n}{2}\ \mathrm{Sin}\ \beta$
C_{36}	$\frac{3n}{4}\ (1 + \cos 2\beta)$
C_{37}	$\frac{9n}{8\ R_{sb}}\ (\frac{7}{6}\ x_o^2 + 2\ y_o^2)\ \mathrm{Sin}\ \beta$
C_{38}	$\frac{15n}{4\ R_{sb}}\ x_o y_o \ \mathrm{Sin}\ 2\beta$
C_{39}	C_{16}
C_{40}	$\frac{15n}{4\ R_{sb}}\ y_o \ \mathrm{Sin}\ 2\beta$
C_{41}	$\frac{21n}{8\ R_{sb}}\ x_o \cos \beta$
C_{42}	$\frac{3n}{4\ R_{sb}}\ y_o$
x_o	$\frac{Q\ (1 - e \cos \mu)}{2\ (1-e)}$
y_o	$\frac{\sqrt{3}}{2}\ \frac{(1 - e \cos \mu)}{(1 - e)}$
Q	$1 - 2m$
R_{sb}	$\frac{a_b\ (1-e_1 \cos \mu_1)}{a_m\ (1 - e_m)}$
F	$1/(1 - e \cos \mu)$
N	M_s/R_{sb}^3

A further simplification of the equations of motion, equations (11) - (13), can be made. Looking at Table I one can see a number of coefficients where the factor $1/R_{sb}$ appears explicitly. For example, typical values of C_2 and C_{14} are .0037 and .000027, respectively. The resultant equations neglecting these terms are

$$\ddot{x} - C_1\dot{y} - (C_2 + C_{10} \operatorname{Cos} 2\phi)_x - (C_3 - C_{11} \operatorname{Sin} 2\phi)_y - (-C_4$$
$$-C_4 \operatorname{Cos} 2\phi)_z = C_5 + C_6 \operatorname{Cos} 2\phi - C_7 \operatorname{Sin} 2\phi \qquad (14)$$

$$\ddot{y} + C_1 \dot{x} - (C_{18} - C_{11} \operatorname{Sin} 2\phi)_x - (C_{19} - C_{25} \operatorname{Cos} 2\phi)_y$$
$$- (C_{31} \operatorname{Sin} 2\phi)_z = C_{20} - C_{21} \operatorname{Cos} 2\phi - C_{22} \operatorname{Sin} 2\phi \qquad (15)$$

$$\ddot{z} - (-C_4 - C_4 \operatorname{Cos} 2\phi)_x - (C_{35} \operatorname{Sin} 2\phi)_y - (C_{32} + C_{36} \operatorname{Cos} 2\phi)_z$$
$$= -C_{33} - C_{33} \operatorname{Cos} 2\phi + C_{34} \operatorname{Sin} 2\phi \qquad (16)$$

where the coefficients are as listed in Table I.

Very Restricted Four Body Problem

Just as the linearized equations of motion for the restricted three body case could be obtained from the three body case, the very restricted four body equations can be obtained from the equations of motion for the four body case (Eqn 14-16). The very restricted four body problem implies that the orbits of the earth and moon about their barycenter and the earth-moon barycenter about the sun are circular (i.e., $e = e_1 = 0$). Another restriction is that all orbits are coplaner (i.e., $\beta = 0$). When these conditions of the very restricted four body problem are imposed, the equations reduce to those developed by deVries (Ref 4:18-19) who started

with the very restricted four body model. Figures 5 and 6 provide a comparison between the trajectory and control determined by the very restricted four body equations (Ref 8) and the trajectory and control as determined by the four body equations (Eqn 14-16) with the lunar and barycenter eccentricity zero ($e = e_1 = 0$) and the inclination of the ecliptic zero ($\beta = 0$). The similarity of the two verifies the four body derivation.

Assumptions

The following is a summarization of the assumptions used in the derivation of the linearized equations of motion about the L_4 point for the four body system of Figure 2.

1. All masses are point masses. The center of mass of the sun is assumed to be the center of mass of the four body solar system.
2. The orbits of the earth, moon and earth-moon barycenter are assumed to be elliptical.
3. The fourth body (satellite) is small enough that it does not affect the motion of the other three massive bodies.
4. The perturbation of the satellite from L_4 is small compared to the distance from L_4 to the earth-moon barycenter.

Normalized Units

Unless otherwise noted all quantities are expressed in normalized units. The astronomical constants upon which the normalizations are based are

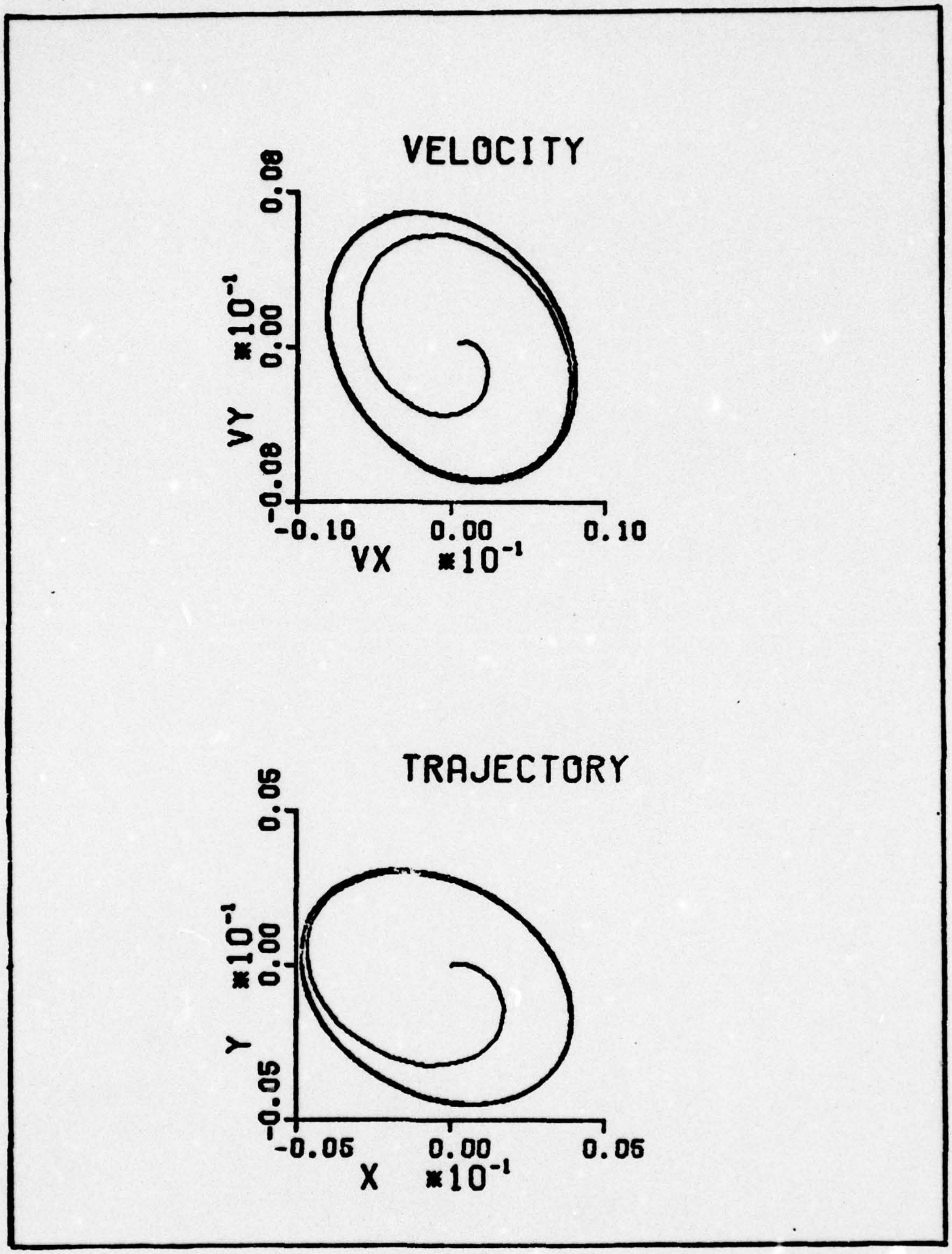


Figure 5

Trajectory - Very Restricted Four Body Equations (Ref 8:38)

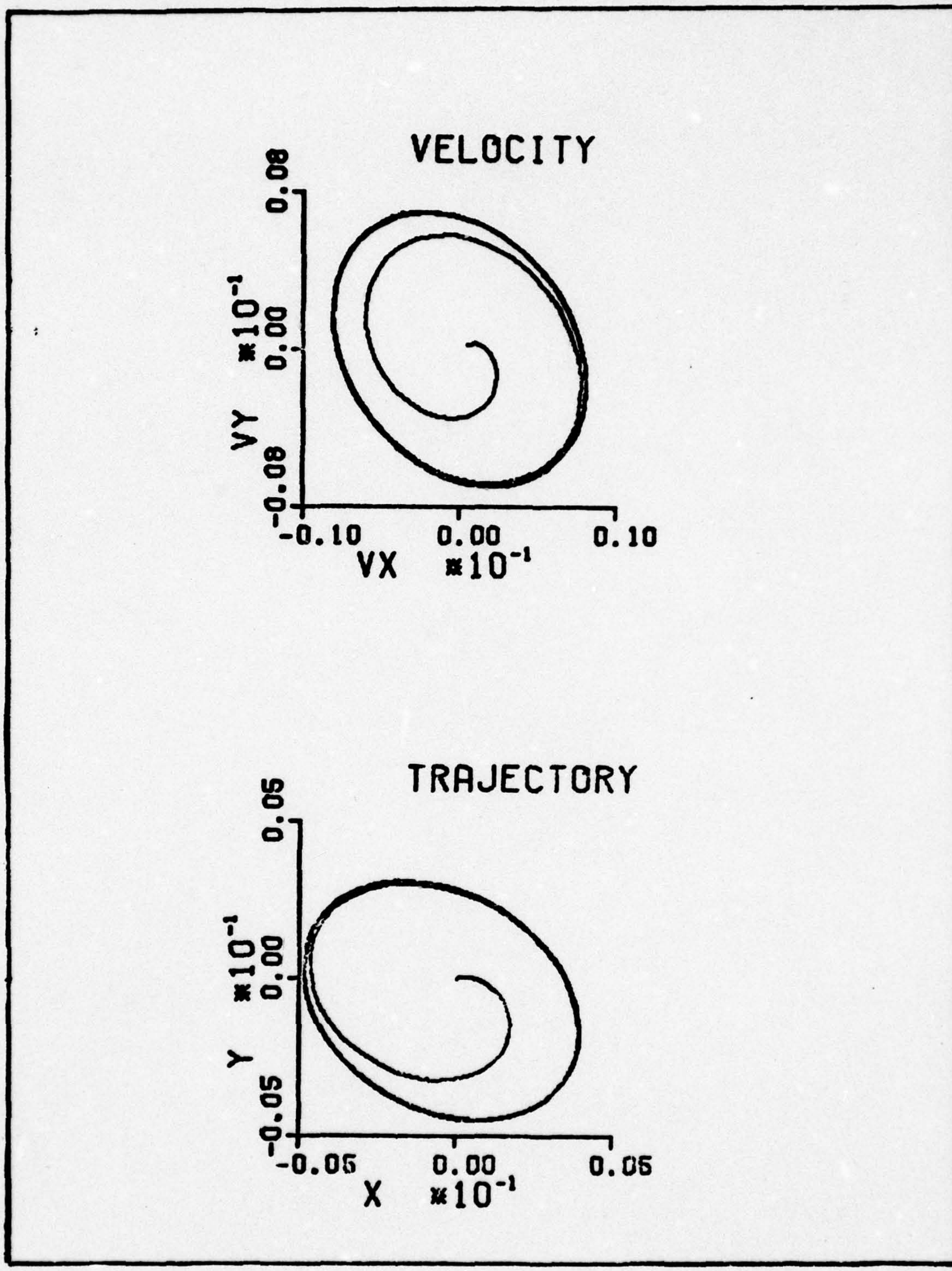


Figure 6

Trajectory - Four Body Equations ($e = e_1 = \beta = 0$)

$1/m = 82.301$

$m_s = 328905.2$

$\Omega = .0748$

$\omega = 1$

$k^2 = 1$

The conversion from normalized units to physical units can be made through the following relationships (Ref 1:429)

1 time unit	=	4.384 mean solar days
1 distance unit	=	384400 kilometers
1 velocity unit	=	1023 meters/second
1 acceleration unit	=	.002724 meters/second^2

Known Station Keeping Costs

In the previous works where the planer problem has been considered, the cost of station keeping has varied from zero (for the uncontrolled satellite (Ref 11:84)) to .008567 normalized acceleration units (for the perfectly controlled satellite (Ref 8:6-7, 29-31)).

The cost for perfect station keeping can be found by using a control which exactly cancels the forcing terms of the equations of motion, (Equations (14) - (16)) i.e.,

$$\begin{aligned} U_x &= -(C_5 + C_6 \cos 2\phi - C_7 \sin 2\phi) \\ U_y &= -(C_{20} - C_{21} \cos 2\phi - C_{22} \sin 2\phi) \\ U_z &= -(-C_{33} - C_{33} \cos 2\phi + C_{34} \sin 2\phi) \end{aligned} \tag{17}$$

The average magnitude of the thrust acceleration is

$$U_{av} = (1/2\pi) \int_0^{2\pi} (U_x^2 + U_y^2 + U_z^2)\, d\phi \tag{18}$$

But for the case at hand the coefficients of the forcing terms of equation (18) are time varying. Considering a constant ω and Ω as in the very restricted four body case the variable of integration can be transformed to dt by the relations

$$\phi = \alpha + (\omega - \Omega)t \; ; \quad d\phi = (\omega - \Omega)\, dt$$

Thus, equation (18) becomes

$$U_{av} = \frac{1}{T} \int_{o}^{T} (U_x^2 + U_y^2 + U_z^2)\, dt \qquad (19)$$

Using the computer to integrate the equation, the average thrust acceleration for zero drift is .008697 normalized acceleration units. (Note that the limits of ϕ from 0-2π correspond to time limits of 0-6.82 normalized time units).

Performance Index

It is useful to define a quantity that will given an indication of the cost of controlling the satellite independent of the vehicle weight or rocket performance. Looking at the rocket equation

$$[\text{Fuel weight}] = [\text{Final weight}]\,[\exp(\Delta V/(g_o I_{sp})-1)] \qquad (20)$$

one can see that minimizing the total velocity increment, ΔV, will minimize the fuel required for a given specific impulse (I_{sp}) and no staging. So, ΔV seems to be an apt quantity to use as a performance index. The total velocity increment, ΔV, can be calculated by

$$\Delta V = \int_0^T \| U(t) \| \, dt \tag{21}$$

where U(t) is the control specified by the quadratic cost function, equation (28).

II. Analysis

Optimal Control Problem

For a linear system, the optimal control with respect to a quadratic cost function can be obtained as a linear combination of the states (Ref 3: 148-151).

Given the linear, time varying system

$$\dot{\underline{X}}(t) = A(t)\ \underline{X}(t) + B(t)\ \underline{U}(t) \tag{22}$$

and the quadratic cost

$$J = \tfrac{1}{2}\ \underline{X}(T)^T\ K_f\ \underline{X}(T) + \tfrac{1}{2}\int_0^t (\underline{X}(t)^T\ Q(t)\ \underline{X}(t) + \underline{U}(t)^T\ R(t)\ U(t))\ dt \tag{23}$$

where K_f and $Q(t)$ are positive semidefinite matrices and $R(t)$ is a positive definite matrix.

The optimal control is given by

$$\underline{U}(t) = -R(t)^{-1}B(t)^T K(t)\ \underline{X}(t) \tag{24}$$

where $K(t)$ is the unique solution to the matrix Riccati equation

$$\dot{K}(t) = -K(t)\ A(t) - A(t)^T\ K(t) + K(t)\ B(t)\ R(t)^{-1}\ B(t)^T\ K(t) - Q(t) \tag{25}$$

subject to the boundary condition $K(T) = K_f$.

Figure 7 gives the form of the optimal system. The state equations (Eq 21) can be written as

$$\dot{\underline{X}}(t) = [A(t) - B(t)\ R(t)^{-1}\ B(t)^T\ K(t)]\ \underline{X}\ (t) \tag{26}$$

For this system the matrices $R(t)$ and $B(t)$ are specified as constants. Thus, the matrix $K(t)$ determines the gain of

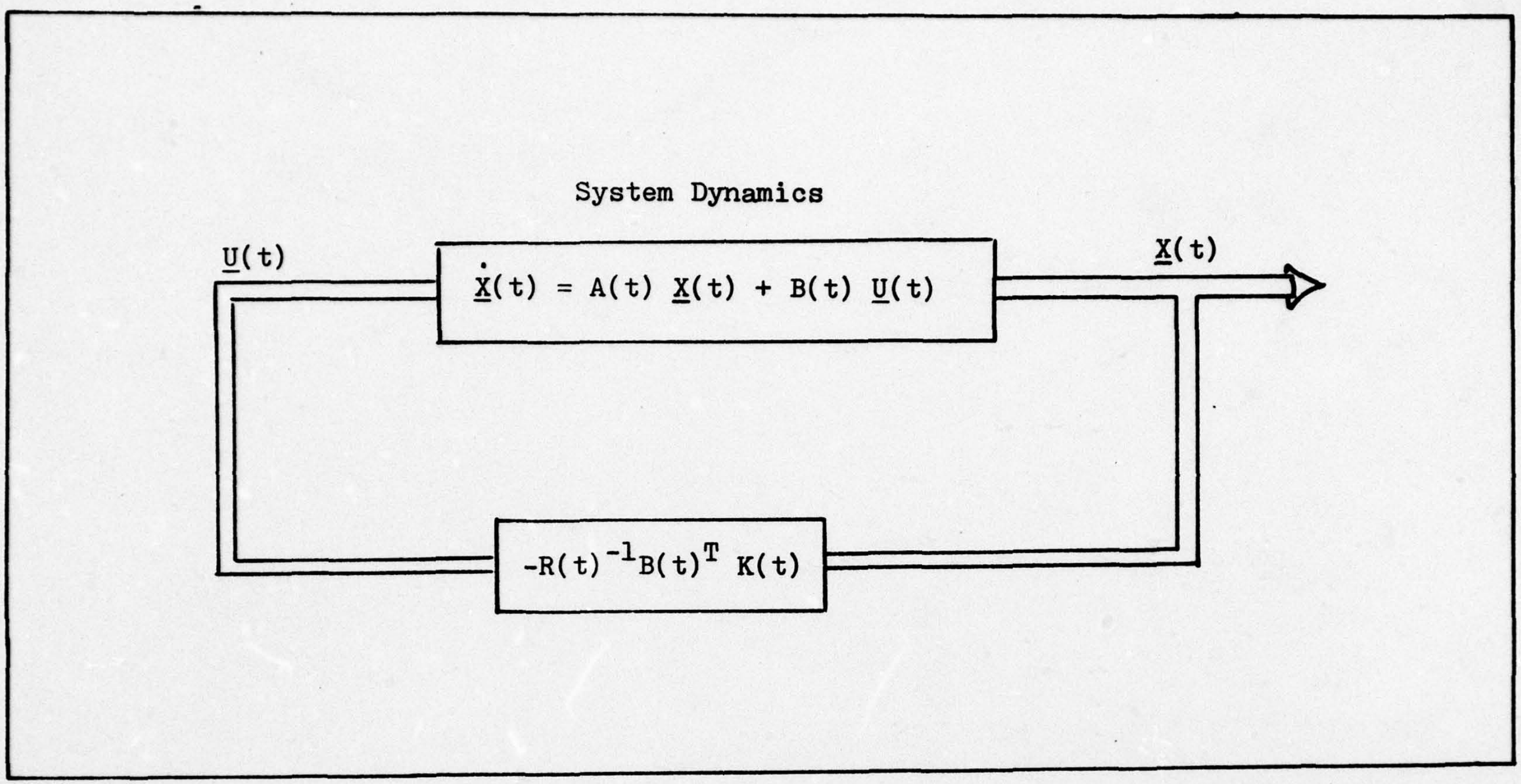


Figure 7

Structure of the Optimal Control (Ref 8)

of the feedback path and the behavior of the system. $K(t)$ will be referred to as the "gain" matrix. As seen from equation (25), it is independent of the states. Once the matrices $A(t)$, $B(t)$, $R(t)$, $Q(t)$, K_f and the final time are specified equation (25) is solved backwards in time to give the time varying gain schedule.

State Equations

The four body equations of motion for the satellite with respect to the L4 point were derived in chapter I and are restated here for convenience as

$$\ddot{x} = C_1 \dot{y} + (C_2 + C_3 \cos 2\phi)x + (C_4 - C_5 \sin 2\phi)y + (-C_{15} - C_{15} \cos 2\phi)z + C_6 + C_7 \cos 2\phi - C_8 \sin 2\phi \quad (14)$$

$$\ddot{y} = -C_1 \dot{x} + (C_9 - C_5 \sin 2\phi)x + (C_{10} - C_{11} \cos 2\phi)y + (C_{19} \sin 2\phi)z + C_{12} - C_{13} \cos 2\phi - C_{14} \sin 2\phi \quad (15)$$

$$\ddot{z} = (-C_{15} - C_{15} \cos 2\phi)x + (C_{16} \sin 2\phi)y + (C_{20} + C_{21} \cos 2\phi)z - C_{17} - C_{17} \cos 2\phi + C_{18} \sin 2\phi \quad (16)$$

where the coefficients have been renumbered and are listed in Table II. The equations of motion can be put in the form of equation (25) by choosing the following states

$$x_1 = x;\ x_2 = \dot{x};\ x_3 = y;\ x_4 = \dot{y};\ x_5 = 1;\ x_6 = z;\ x_7 = \dot{z};$$
$$x_8 = \cos 2\phi;\ x_9 = \sin 2\phi$$

The state equations are

Table II

Coefficients of the State Equations

Term	Expression
C_1	$2\sqrt{1-e^2}\ F^2$
C_2	$(1-e^2)F^4 - \frac{1}{4}F^3 + \frac{N}{4}(3 \cos 2\beta - 1)$
C_3	$\frac{3}{4} N (1 + \cos 2\beta)$
C_4	$\frac{3\sqrt{3}}{4} QF^3 - 2e \text{ Sin } \mu\sqrt{1-e^2}\ F^4$
C_5	$\frac{3}{2} N \cos \beta$
C_6	$\frac{N}{4} x_o (3 \cos 2\beta - 1)$
C_7	$\frac{3}{4} N x_o (1 + \cos 2\beta)$
C_8	$\frac{3}{2} N y_o \cos \beta$
C_9	$\frac{3\sqrt{3}}{4} QF^3 + 2e \sin \mu\sqrt{1-e^2}\ F^4$
C_{10}	$(1-e^2)F^4 + \frac{5}{4}F^3 + \frac{N}{2}$
C_{11}	$\frac{3N}{2}$
C_{12}	$\frac{N}{2} y_o$
C_{13}	$\frac{3N}{2} y_o$
C_{14}	$\frac{3N}{2} x_o \cos \beta$

Table II - continued

Term	Expression
C_{15}	$\frac{3}{4}$ N Sin 2β
C_{16}	$\frac{3}{2}$ N Sin β
C_{17}	$\frac{3}{4}$ N x_o Sin 2β
C_{18}	$\frac{3}{2}$ N y_o Sin 2β
C_{19}	$\frac{3}{4}$ N Sin β
C_{20}	$-F^3 - \frac{N}{4} (1 + 3 \cos 2\beta)$
C_{21}	$\frac{3N}{4} (1 + 1 \cos 2\beta)$
x_o	$\frac{Q (1 - e \cos \mu)}{2 (1-e)}$
y_o	$\frac{\sqrt{3}}{2} \frac{(1 - e \cos \mu)}{(1-e)}$
Q	1 - 2m
R_{sb}	$\frac{a_b (1 - e_b \cos \mu_b)}{a_m (1 - e)}$
F	$1/(1 - e \cos \mu)$
N	M_s/R^3_{sb}

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = (C_2 + C_3 \cos 2\phi)x_1 + (C_4 - C_5 \sin 2\phi)x_3 + C_1x_4 + C_6x_5$$
$$+ (-C_{15} - C_{15} \cos 2\phi)x_6 + C_7x_8 - C_8x_9$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = (C_9 - C_5 \sin 2\phi)x_1 - C_2x_2 + (C_{10} - C_{11} \cos 2\phi)x_3$$
$$+ C_{12}\, x_5 + C_{19} \sin 2\phi\, x_6 - C_{13}x_8 - C_{14}x_9$$

$$\dot{x}_5 = 0 \tag{27}$$

$$\dot{x}_6 = x_7$$

$$\dot{x}_7 = (-C_{15} - C_{15} \cos 2\phi)x_1 + C_{16} \sin 2\phi\, x_3 - C_{17}\, x_5$$
$$+ (C_{20} + C_{21} \cos 2\phi)x_6 - C_{17}x_8 + C_{12}x_9$$

$$\dot{x}_8 = -2\omega\, x_9$$

$$\dot{x}_9 = 2\omega\, x_8$$

The object is to minimize drift and fuel used; thus, the unconstrained quadratic cost function used is

$$J = \tfrac{1}{2} \int_o^t [q\,(x_1^2 + x_3^2 + x_6^2) + (U_1^2 + U_2^2 + U_3^2)]\, dt \tag{28}$$

This is in the form of equation (23), where

$$K_f = [0] \quad \text{(a 9x9 zero matrix)}$$

$$Q(t) = \begin{vmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

Note that the states, x_1, x_3, x_7, are weighted by q.

$$R(t) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad U(t) = \begin{vmatrix} U_1 \\ U_2 \\ U_3 \end{vmatrix}$$

$$B(t) = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad t_0 = 0$$

$$A(t) = \begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_2 + C_3 \cos 2\phi & 0 & C_4 - C_5 \sin 2\phi & C_1 & C_6 & -C_{15} - C_{15} \cos 2\phi & 0 & C_7 & -C_8 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ C_9 - C_5 \sin 2\phi & -C_1 & C_{10} - C_{11} \cos 2\phi & 0 & C_{12} & C_{19} \sin 2\phi & 0 & C_{13} & C_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -C_{15} - C_{15} \cos 2\phi & 0 & C_{16} \sin 2\phi & 0 & -C_{17} & C_{20} + C_{21} \cos 2\phi & 0 & -C_{17} & C_{18} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2w \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2w & 0 \end{vmatrix}$$

Using the matrices given above and performing the indicated operations, Eqn (24) and (22) can be simplified to

$$\underline{U}(t) = - \begin{vmatrix} k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} & k_{27} & k_{28} & k_{29} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} & k_{47} & k_{48} & k_{49} \\ k_{71} & k_{72} & k_{73} & k_{74} & k_{75} & k_{76} & k_{77} & k_{78} & k_{79} \end{vmatrix} \underline{X}(t) \qquad (29)$$

$$\dot{\underline{X}}(t) = \left| A(t) - \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} & k_{27} & k_{28} & k_{29} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} & k_{47} & k_{48} & k_{49} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_{71} & k_{72} & k_{73} & k_{74} & k_{75} & k_{76} & k_{77} & k_{78} & k_{79} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} \right| X(t) \qquad (30)$$

III. Gain Matrix Computations

Optimal Gains

The gain matrix, K(t), is computed by solving the matrix Riccati equation (Eq 25) backwards from the final time (T). The solution is optimal gain schedule based upon the specified final time. This gain schedule would then be used to control the satellite as specified by the control equation

$$\underline{U}(t) = -R(t)^{-1}\, B(t)^{T}\, K(t)\, \underline{X}(t) \tag{24}$$

Figures 8 - 10 present plots of typical gain elements versus time for a variety of weights, q, (Ref Equation (28)). As seen in the plots all exhibit common factors. Each has a transitory phase and a steady-state phase that is periodic with approximately the lunar period. As the weighting, q, is increased, the transitory phase shortens, and the peak amplitude in the transient phase decreases. The period remains unchanged for the particular gain element.

In the matrix of coefficients of the states, the "A" matrix, there is a dependance upon the eccentric anomalies of the earth-moon barycenter and the moon. This leads to the question of what effect the relative positions of the earth, moon and sun have upon the computations. Figures 11-13 depict gain elements computed for q = 1 for a variety of initial conditions. The changes from one plot to another are quite small, but the effects on system performance are significant. Chapter IV will examine the effects of initial conditioning in detail.

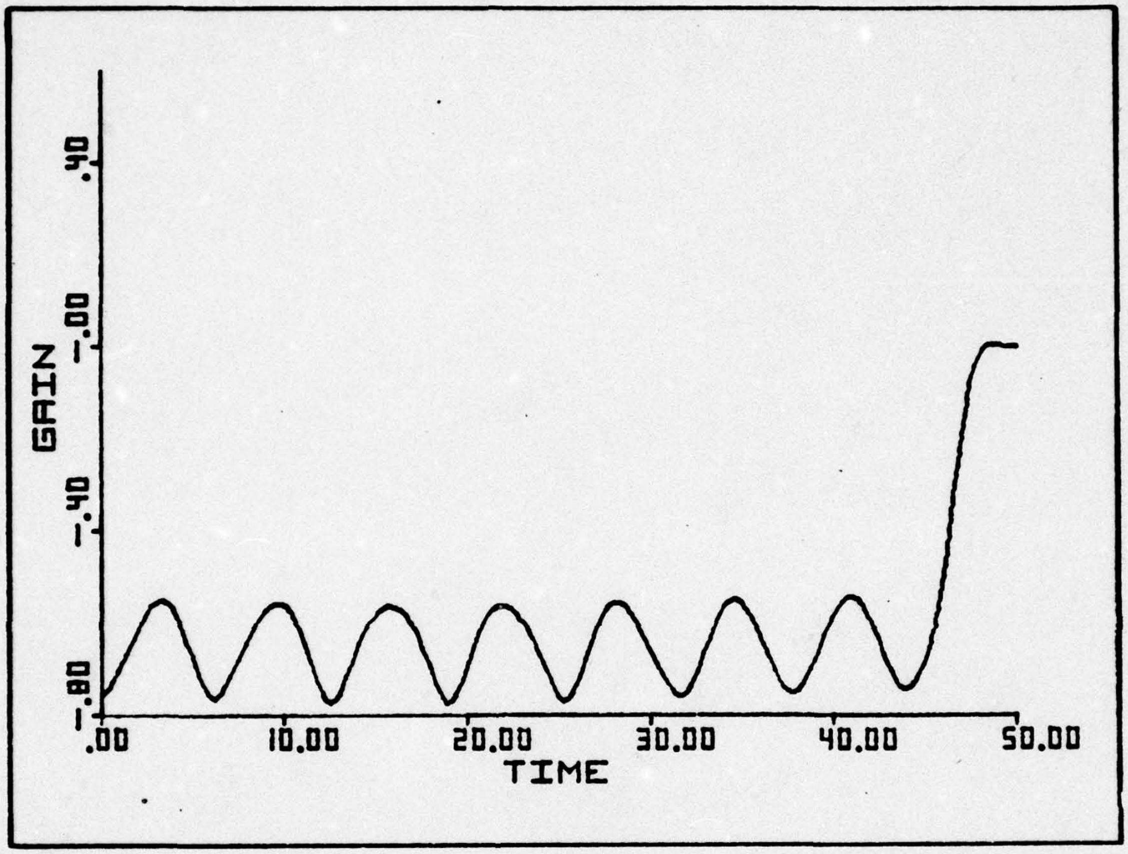


Figure 8

Gain Element Vs Time

(q = .01)

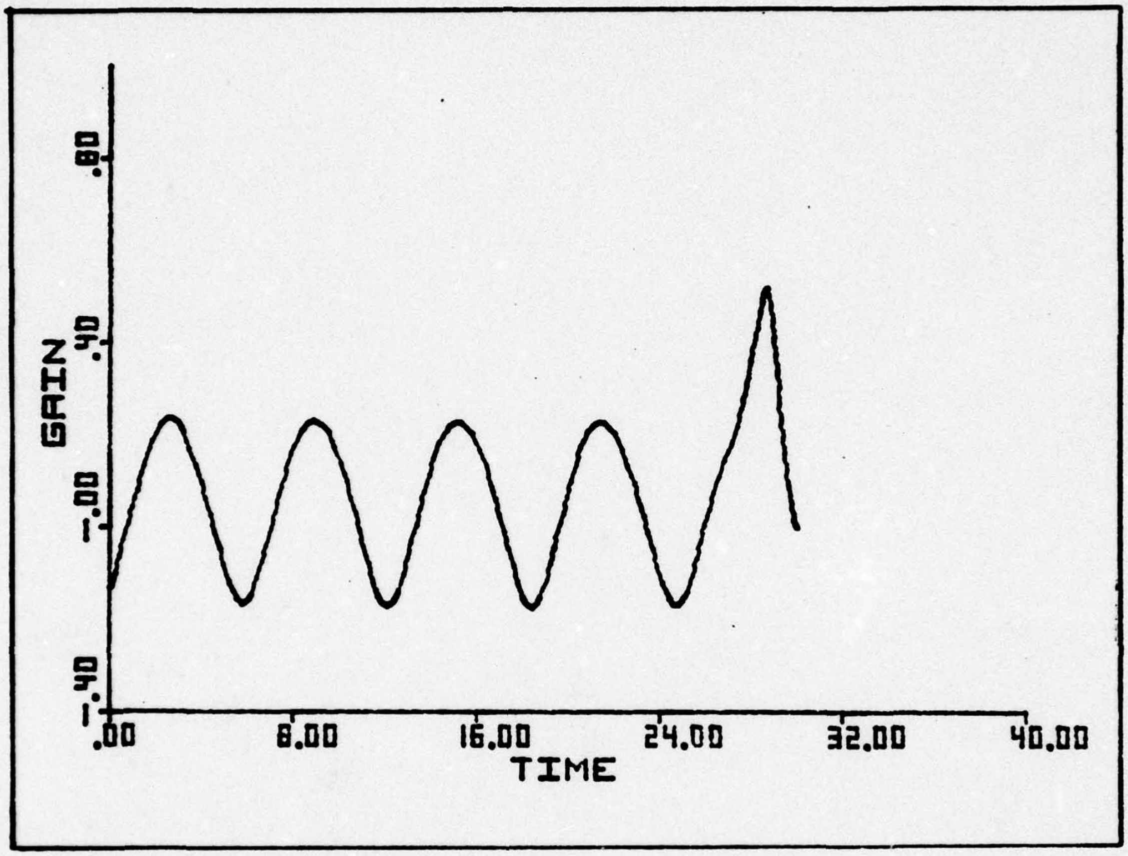


Figure 9

Gain Element Vs Time

(q = 1)

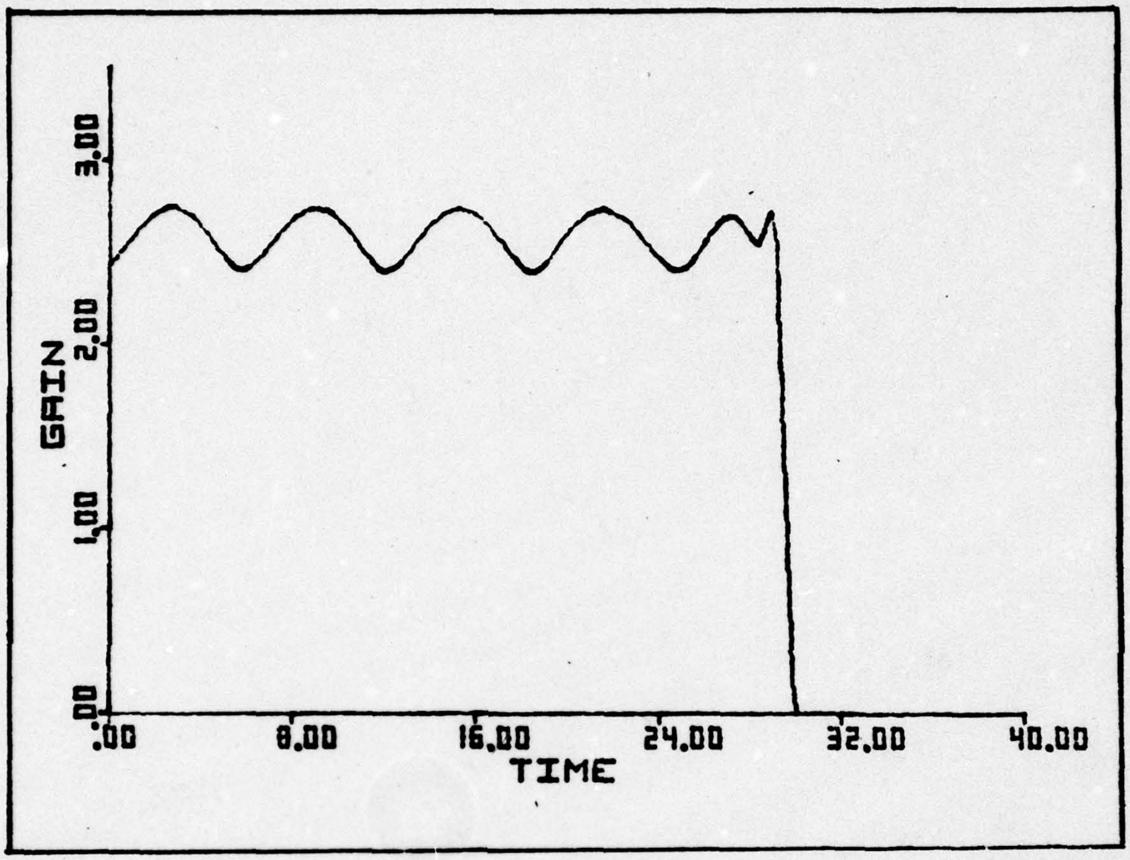


Figure 10

Gain Element Vs Time

(q = 10)

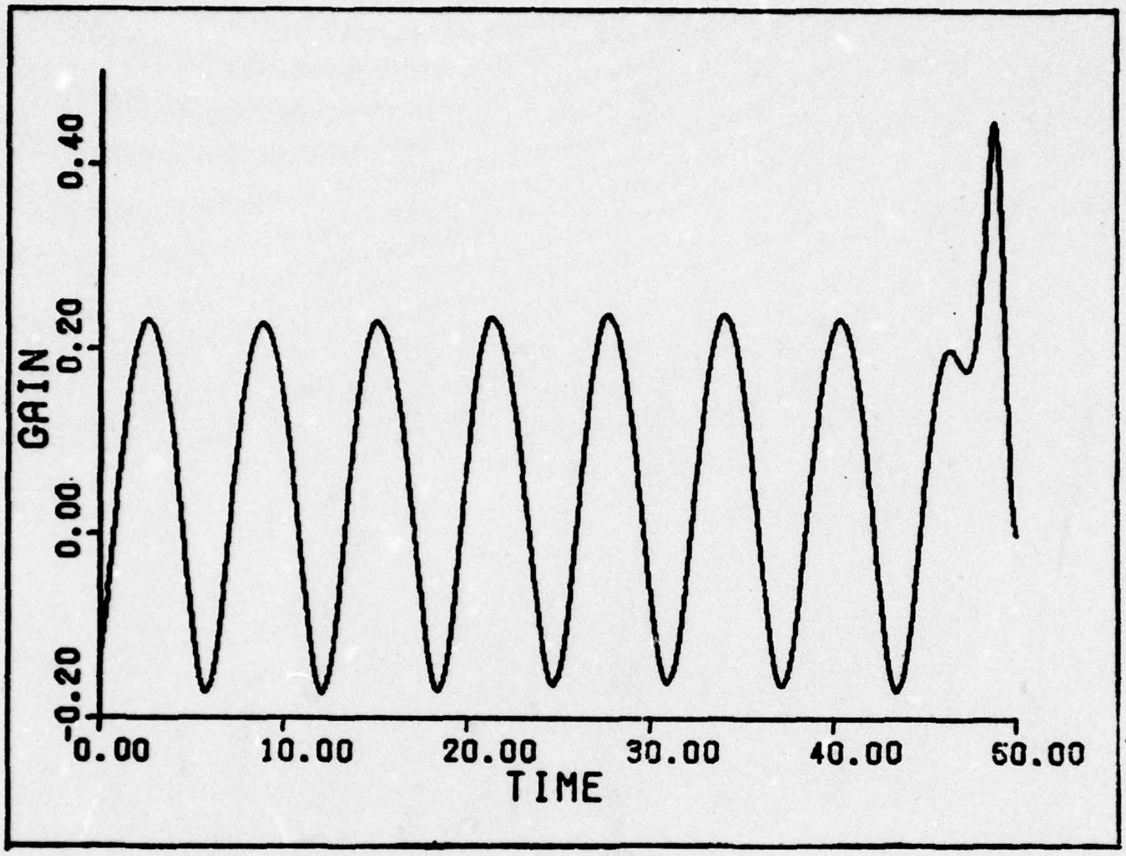


Figure 11

Gain Element Vs Time

($q = 1$; $\alpha = 30^{\circ}$)

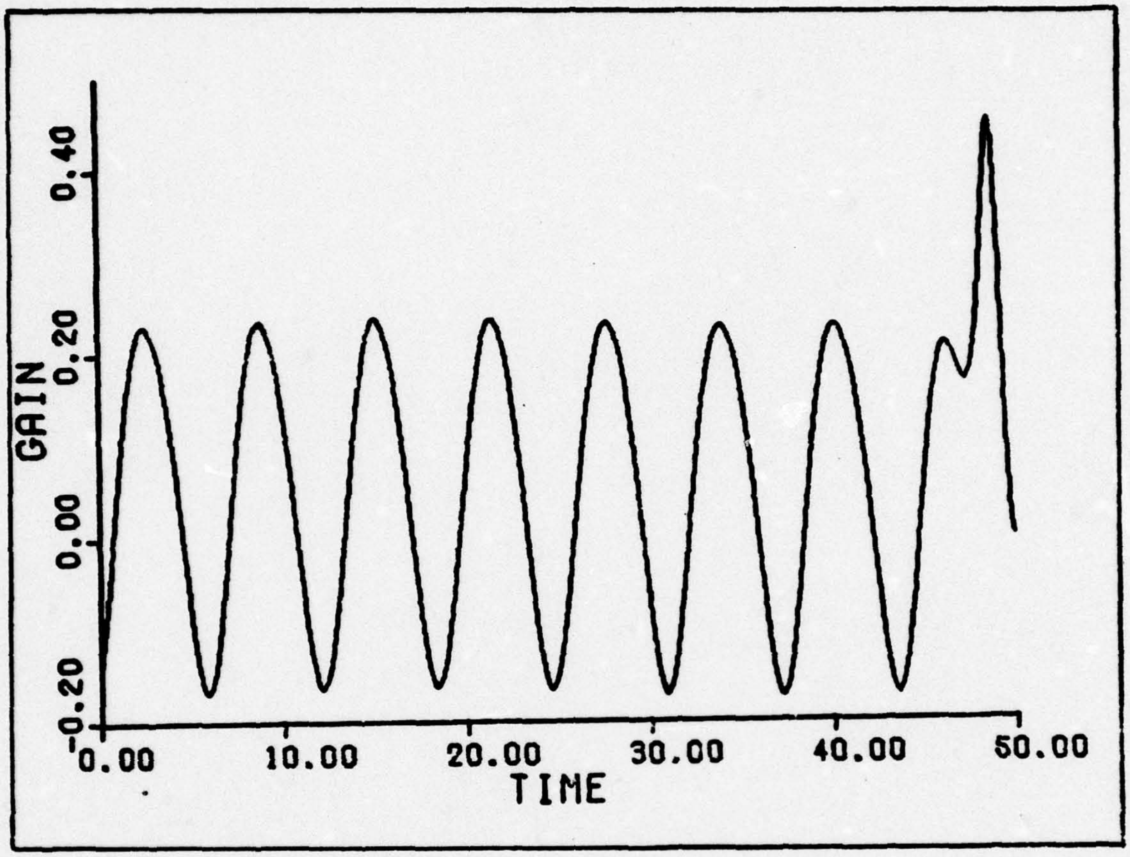


Figure 12

Gain Element Vs Time

($q = 1$; $\alpha = 90^{\circ}$)

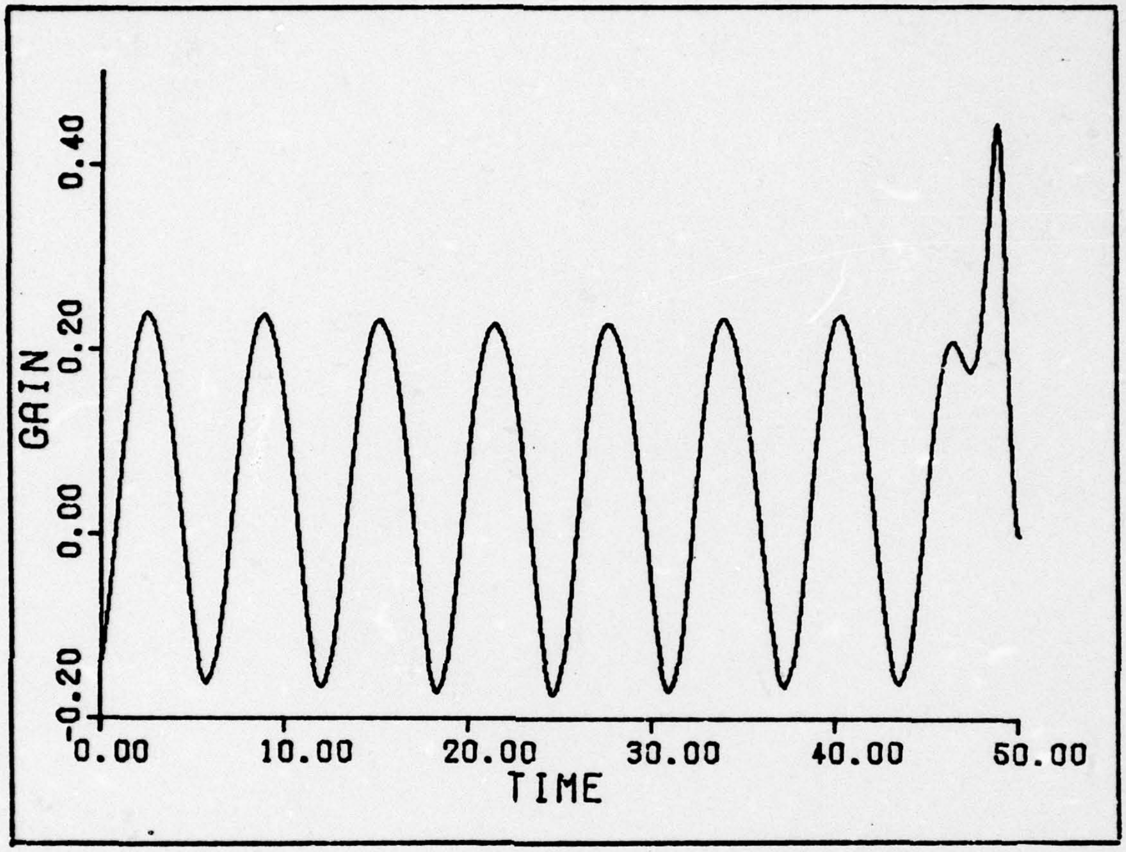


Figure 13

Gain Element Vs Time

($q = 1$; $\alpha = 150^{o}$)

Constant Gain Elements

While the varying gain schedule above gives the optimum path and control for the satellite, it is a very costly method in terms of computation and storage requirements and so, there is motivation to derive a less cumbersome gain schedule. Since the gains are periodic in the steady-state, averaging the gain over the steady-state interval should provide a near optimal constant gain. The use of a constant gain has several advantages. First, there is no requirement for a complete solution to the Riccati equation. It is only necessary for the Riccati equation to be solved for a short period so that the constant gain can be computed; thus, there is no dependence on final time as there is for the optimal gain. Second, the only storage required is for a single gain matrix rather than an entire gain history. It must be noted, however, that the price for the simplicity of the constant gain is suboptimal station keeping. Table III lists the constant gains computed for a variety of weights with $\alpha = 0$. As the weighting, q, increases the values of the gain increase. In terms of performance it will be shown in Chapter IV that this increase in gain values will result in a smaller variance from the L4 position and an increase in cost. Table IV shows the variance of the gains as the initial sun direction, α, is varied.

The computer programs used to compute the fixed and varying gains are listed in Appendix B.

Table III

Fixed Gains

q	K_{21}	K_{22}	K_{23}
.01	-6.6937E-01	5.5528E-01	-1.1335E+00
.02	-7.3003E-01	6.0598E-01	-1.2463E+00
.03	-7.4116E-01	6.3190E-01	-1.2954E+00
.10	-7.0746E-01	7.1021E-01	-1.3585E+00
.30	-4.9936E-01	8.2081E-01	-1.3173E+00
1.00	3.2440E-02	1.0738E+00	-1.2661E+00
3.00	9.2753E-01	1.4897E+00	-1.2288E+00
10.00	2.5732E+00	2.1918E+00	-1.5171E+00
100.00	9.6678E+00	4.3105E+00	-3.2773E+00
1.E+04	9.9749E+01	1.4112E+01	-1.2855E+01

q	K_{41}	K_{42}	K_{43}
.01	8.2675E-01	-4.5763E-01	1.2747E+00
.02	1.0583E+00	-5.1332E-01	1.5912E+00
.03	1.2093E+00	-5.4028E-01	1.7877E+00
.10	1.7251E+00	-5.4847E-01	2.3828E+00
.30	2.2759E+00	-4.6732E-01	2.9463E+00
1.00	2.9310E+00	-2.8703E-01	3.5581E+00
3.00	3.5645E+00	-8.7659E-02	4.1924E+00
10.00	4.2697E+00	8.6078E-02	5.3132E+00
100.00	6.0692E+00	1.7286E-01	1.1592E+01
1.E+04	1.5434E+01	8.1488E-02	1.0130E+02

q	K_{71}	K_{72}	K_{73}
.01	2.0049E-04	-2.3693E-04	4.1878E-04
.02	1.6390E-04	-2.3318E-04	3.9702E-04
.03	1.2442E-04	-2.2747E-04	3.6286E-04
.10	-4.7705E-05	-2.1019E-04	2.1300E-04
.30	-2.1486E-04	-2.0717E-04	8.1473E-05
1.00	-3.8574E-04	-2.1272E-04	2.2487E-06
3.00	-4.9425E-04	-2.0856E-04	1.2445E-05
10.00	-5.5786E-04	-1.8841E-04	1.7092E-06
100.00	-5.9674E-04	-1.2623E-04	2.7829E-05
1.E+04	-6.3627E-04	-4.4773E-05	4.8417E-05

Table III

continued

q	K_{24}	K_{25}	K_{26}
.01	-4.5763E-01	-1.1087E-03	-2.3421E-04
.02	-5.1332E-01	-1.2109E-03	-2.3417E-04
.03	-5.4028E-01	-1.2509E-03	-2.3291E-04
.10	-5.4847E-01	-1.2619E-03	-2.2460E-04
.30	-4.6732E-01	-1.0937E-03	-2.3102E-04
1.00	-2.8703E-01	-6.5882E-04	-2.6752E-04
3.00	-8.7659E-02	-7.3255E-05	-3.4941E-04
10.00	8.6078E-02	6.0977E-04	-4.4524E-04
100.00	1.7786E-01	1.2218E-03	-5.5599E-04
1.E+04	8.1488E-02	1.2301E-03	-6.3265E-04

q	K_{44}	K_{45}	K_{46}
.01	6.8330E-01	1.2680E-03	2.7118E-04
.02	9.0443E-01	1.5905E-03	3.0631E-04
.03	1.0466E+00	1.7927E-03	3.3044E-04
.10	1.4969E+00	2.4331E-03	3.0784E-04
.30	1.9446E+00	3.0311E-03	2.0121E-04
1.00	2.4173E+00	3.6008E-03	5.5842E-05
3.00	2.8255E+00	3.8814E-03	-4.6689E-05
10.00	3.3068E+00	3.7539E-03	-9.5309E-05
100.00	4.8853E+00	2.8743E-03	-7.4269E-05
1.E+04	1.4246E+01	2.2575E-03	-1.0638E-05

q	K_{74}	K_{75}	K_{76}
.01	8.2222E-05	-5.7368E-06	3.8978E-03
.02	7.3497E-05	-7.8280E-06	8.4148E-03
.03	5.5020E-05	-9.4330E-06	1.3322E-02
.10	-3.2374E-05	-1.8953E-05	4.5687E-02
.30	-1.0709E-04	-4.2610E-05	1.3455E-01
1.00	-1.3931E-04	-9.6705E-05	4.0404E-01
3.00	-1.2473E-04	-1.6350E-04	9.9080E-01
10.00	-8.8751E-05	-2.2822E-04	2.3027E+00
100.00	-3.2923E-05	-2.9453E-04	9.0302E+00
1.E+04	-1.6929E-06	-3.2710E-04	9.8997E+01

Table III

continued

q	K_{27}	K_{28}	K_{29}
.01	-2.3693E-04	-6.2699E-04	-3.7076E-03
.02	-2.3318E-04	-4.9653E-04	-4.2320E-03
.03	-2.2747E-04	-4.3649E-04	-4.5232E-03
.10	-2.1019E-04	-3.3435E-04	-5.4612E-03
.30	-2.0717E-04	-1.7917E-04	-6.7004E-03
1.00	-2.1272E-04	7.6501E-04	-8.1930E-03
3.00	-2.0856E-04	2.4143E-03	-8.6067E-03
10.00	-1.8841E-04	3.2614E-03	-7.8685E-03
100.00	-1.2623E-04	3.6431E-03	-6.5406E-03
1.E+04	-4.4773E-05	3.6056E-03	-6.5789E-03

q	K_{47}	K_{48}	K_{49}
.01	8.2222E-05	-1.7533E-03	4.1331E-03
.02	7.3497E-05	-2.9406E-03	4.8988E-03
.03	5.5020E-05	-3.8151E-03	5.2554E-03
.10	-3.2374E-05	-7.0729E-03	5.4280E-03
.30	-1.0709E-04	-1.0074E-02	3.8353E-03
1.00	-1.3931E-04	-1.1522E-02	5.8308E-04
3.00	-1.2473E-04	-1.0809E-02	-1.7835E-03
10.00	-8.8751E-05	-9.6166E-03	-2.8655E-03
100.00	-3.2923E-05	-8.1452E-03	-3.8116E-03
1.E+04	-1.6929E-06	-6.7012E-03	-3.8187E-03

q	K_{77}	K_{78}	K_{79}
.01	8.2225E-02	3.6654E-05	2.3998E-05
.02	1.2755E-01	5.6728E-05	3.7283E-05
.03	1.6180E-01	6.9790E-05	4.7900E-05
.10	3.0906E-01	1.2434E-04	9.7997E-05
.30	5.2615E-01	1.9016E-04	1.8605E-04
1.00	9.0487E-01	2.5037E-04	3.6440E-04
3.00	1.4076E+00	2.1711E-04	6.9011E-04
10.00	2.1458E+00	3.3114E-05	7.7265E-04
100.00	4.2497E+00	-2.7223E-04	7.3097E-04
1.E+04	1.4071E+01	-3.3442E-04	6.0479E-04

Table IV

Fixed Gains (Varying α)

	q	K_{21}	K_{22}	K_{23}	K_{24}	K_{25}
0°	1.0	3.2440E-02	1.0738E+00	-1.2261E+00	-2.8703E-01	-6.5882E-04
30°	1.0	3.2595E-02	1.0742E+00	-1.2257E+00	-2.7962E-01	-6.4677E-04
90°	1.0	3.3047E-02	1.0742E+00	-1.2257E+00	-2.8656E-01	-6.3428E-04
120°	1.0	3.3034E-02	1.0747E+00	-1.2261E+00	-2.8670E-01	-6.3689E-04
270°	1.0	3.3299E-02	1.0743E+00	-1.2256E+00	-2.8646E-01	-6.8231E-04
330°	1.0	3.2711E-02	1.0735E+00	-1.2265E+00	-2.8702E-01	-6.7107E-04
	q	K_{41}	K_{42}	K_{43}	K_{44}	K_{45}
0°	1.0	2.9310E+00	-2.8703E-01	3.5581E+00	2.4173E+00	3.6008E-03
30°	1.0	2.9308E+00	-2.7962E-01	3.5577E+00	2.4172E+00	3.5388E-03
90°	1.0	2.9562E+00	-2.8656E-01	3.5566E+00	2.4168E+00	3.4509E-03
120°	1.0	2.9299E+00	-2.8670E-01	3.5566E+00	2.4168E+00	3.4578E-03
270°	1.0	2.9560E+00	-2.8646E-01	3.5567E+00	2.4168E+00	3.7217E-03
330°	1.0	2.9304E+00	-2.8702E-01	3.5577E+00	2.4171E+00	3.6621E-03
	q	K_{71}	K_{72}	K_{73}	K_{74}	K_{75}
0°	1.0	-3.8574E-04	-2.1272E-04	2.2847E-06	-1.3931E-04	-9.6705E-05
30°	1.0	-3.9774E-04	-2.2138E-04	2.0102E-05	-1.3756E-04	-9.4849E-05
90°	1.0	-4.0766E-04	-2.2906E-04	2.2454E-06	-1.3412E-04	-9.2216E-05
120°	1.0	-3.8951E-04	-2.2878E-04	2.5019E-05	-1.3362E-04	-9.2094E-05
270°	1.0	-4.3608E-04	-2.4495E-04	2.9183E-06	-1.4348E-04	-9.9003E-05
330°	1.0	-4.1026E-04	-2.2776E-04	2.5485E-05	-1.2917E-04	-9.8258E-05

Table IV

continued

	q	K_{26}	K_{27}	K_{28}	K_{29}
0°	1.0	-2.6752E-04	-2.1272E-04	7.6501E-04	-8.1930E-03
30°	1.0	-2.6298E-04	-2.2188E-04	7.4897E-04	-8.0412E-03
90°	1.0	-2.8784E-04	-2.2906E-04	7.3002E-04	-7.8743E-03
120°	1.0	-2.8717E-04	-2.2878E-04	7.3059E-04	-7.8991E-03
270°	1.0	-3.0772E-04	-2.4495E-04	7.8494E-04	-8.4671E-03
330°	1.0	-2.8650E-04	-2.2776E-04	8.3505E-04	-8.3407E-03
	q	K_{46}	K_{47}	K_{48}	K_{49}
0°	1.0	5.5842E-05	-1.3931E-04	-1.1522E-02	5.8308E-04
30°	1.0	4.0664E-05	-1.3756E-04	-1.1303E-02	5.7285E-04
90°	1.0	3.1525E-05	-1.3412E-04	-1.1063E-02	5.3098E-04
120°	1.0	4.5652E-05	-1.3362E-04	-1.1089E-02	5.3147E-04
270°	1.0	3.4303E-05	-1.4348E-04	-1.1893E-02	5.7091E-04
330°	1.0	5.6817E-05	-1.2917E-04	-1.1725E-02	5.9171E-04
	q	K_{76}	K_{77}	K_{78}	K_{79}
0°	1.0	4.0404E-01	9.0487E-01	2.5037E-04	3.6640E-04
30°	1.0	4.0409E-01	9.0489E-01	2.4699E-04	3.5697E-04
90°	1.0	4.0417E-01	9.0497E-01	2.4092E-04	3.4978E-04
120°	1.0	4.0415E-01	9.0495E-01	2.4166E-04	3.5049E-04
270°	1.0	4.0402E-01	9.0483E-01	2.5928E-04	3.7624E-04
330°	1.0	4.0405E-01	9.0485E-01	2.5487E-04	3.7079E-04

IV. System Performance

Optimum versus Modified Control

To make the station keeping costs of value to the user, system performance must be evaluated in terms of cost versus distance from the L4 point. In chapter III both constant and time varying gains were computed. These lead to two types of controllers: the optimum controller, using the varying gain schedule; and the modified controller, using the constant gain. Table V gives a comparison of the optimum versus modified controller for q = 1. (Note: Quadratic and ΔV costs are computed by equations (20) and (21) respectively)

Table V

System Comparison

Final Time	System	Peak Drift	ΔV Cost	Quadratic Cost
15	Optimal	.0046075	.064284	.00093150
15	Modified	.0050679	.066905	.00092397
25	Optimal	.0046075	.103100	.0015483
25	Modified	.0050679	.10589	.0015445
45	Optimal	.0046075	.17738	.0026726
45	Modified	.0050679	.17635	.0027804

If ΔV cost were the only consideration, the optimal controller would be the choice. However, when one considers computational requirements, the modified controller is better in that it provides near optimal performance with a fraction of the computational requirements of the optimal controller. Table V shows that as time increases the actual costs of the two systems become nearly equal while the difference in peak

drift is constant. Figures 14 and 15 present graphically the satellite path versus time for the optimal and modified control systems respectively.

One can note from the figures that both controllers exhibit a transient period starting at x=y=z=0. The optimal controller drives the satellite to the steady state in approximately 2.5 time units; the modified controller takes approximately 5 time units to reach steady state.

Constant Gain Performance

For the remainder of this study, the constant gain (modified) controller will be examined. The results will hold for the optimal controller as well. Appendix C lists the computer program used for either constant gain or optimal gain schedules.

In chapter II it was noted that there were two extremes of cost: perfect station keeping and no control. But what of the region between the two extremes? Figure 16 depicts the path of the satellite for an approximate one-year mission when no control is used. The resultant path is unstable. Figures 17 - 19 depict the path for an approximate one-year mission when various weights, q, are used. In all cases, after a brief transient period the satellite enters roughly a steady-state path and remains locked in that path as long as the controller is operational. Several characteristics can be noted from the figures. First, as q is increased the duration of the transient phase is shortened. Second, the steady-state path becomes tighter with increasing q. Fig-

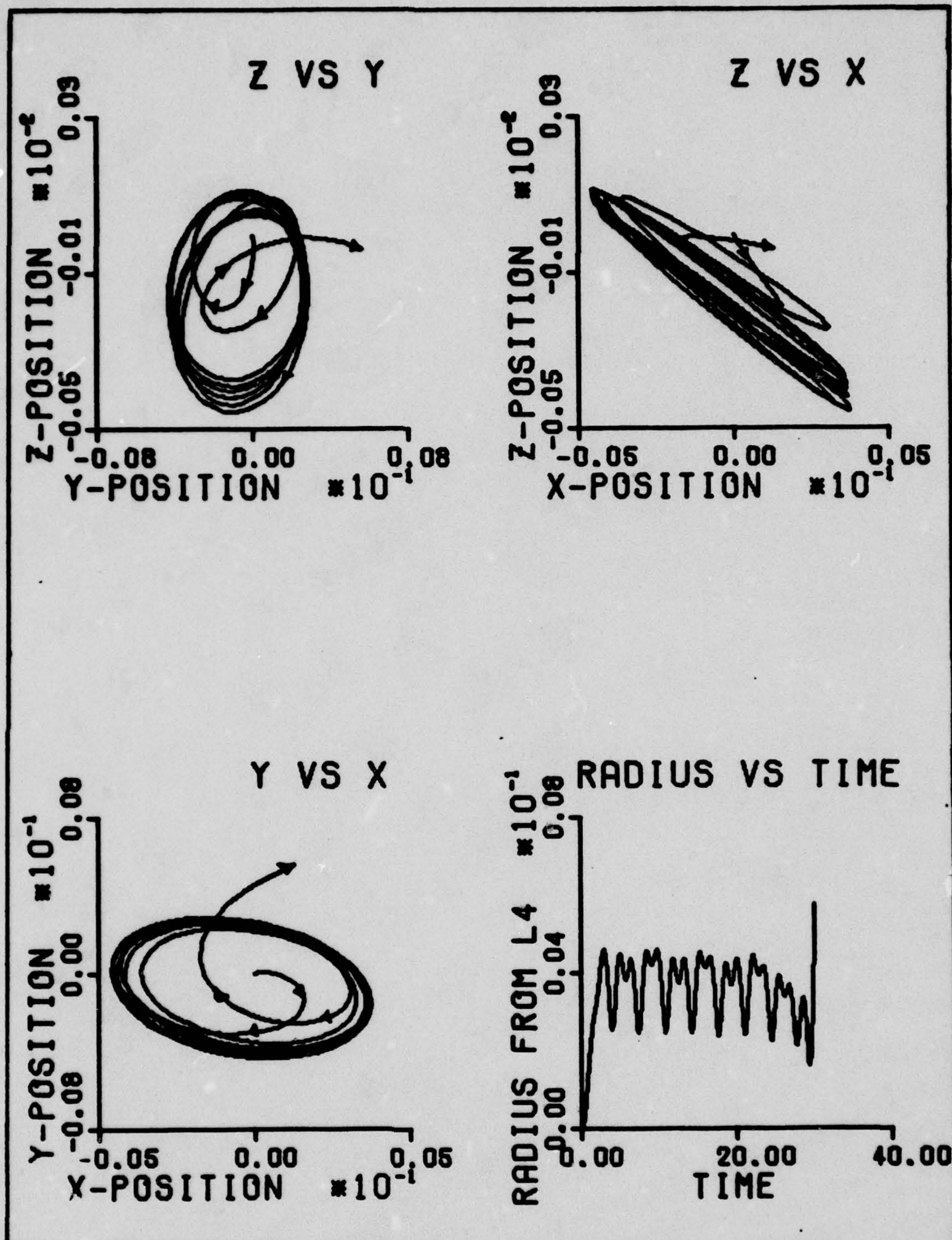


Figure 14

Satellite Trajectory - Optimal Control

($q = 1$; $\alpha = 0^0$)

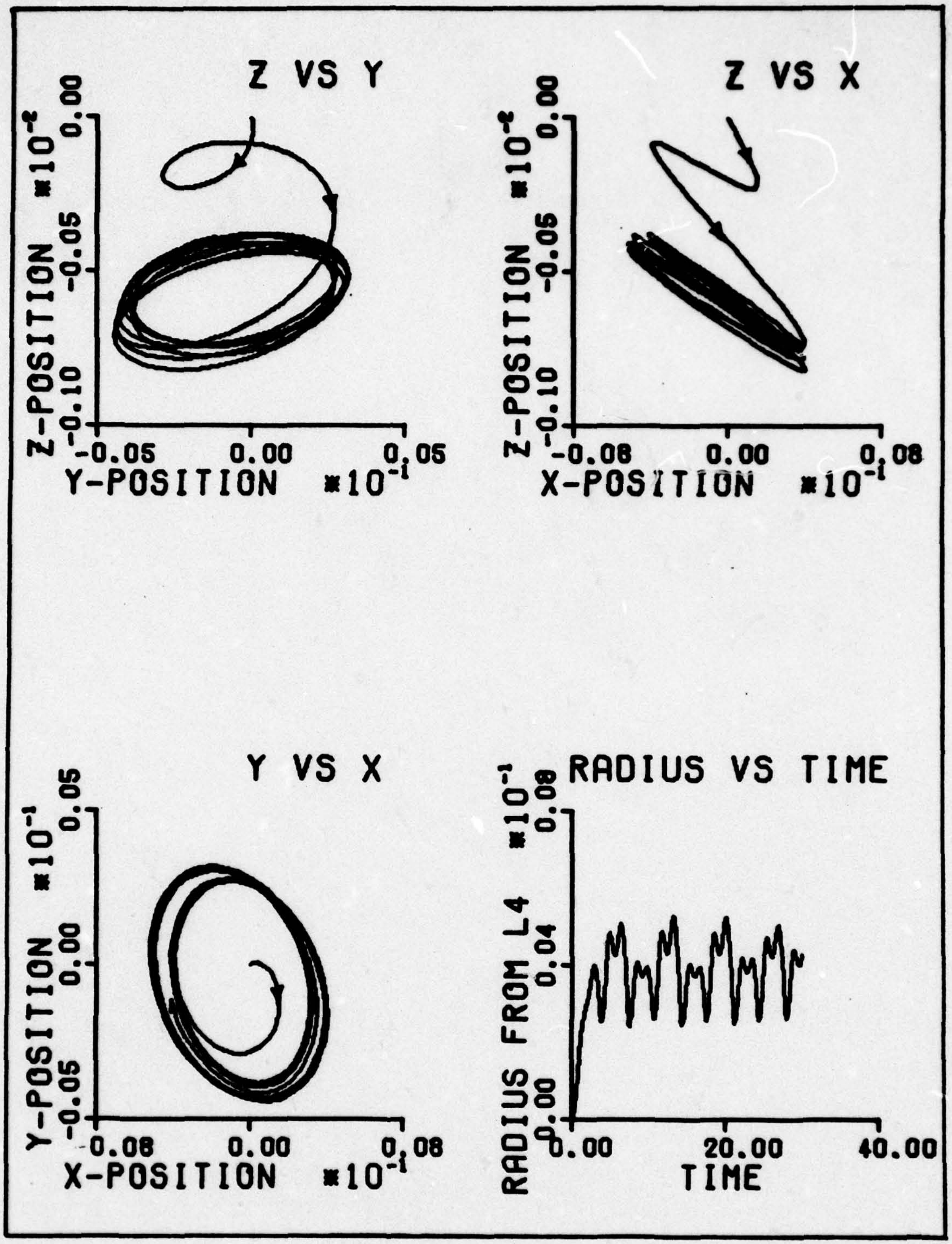


Figure 15

Satellite Trajectory - Modified Control

($q = 1$; $\alpha = 0^{\circ}$)

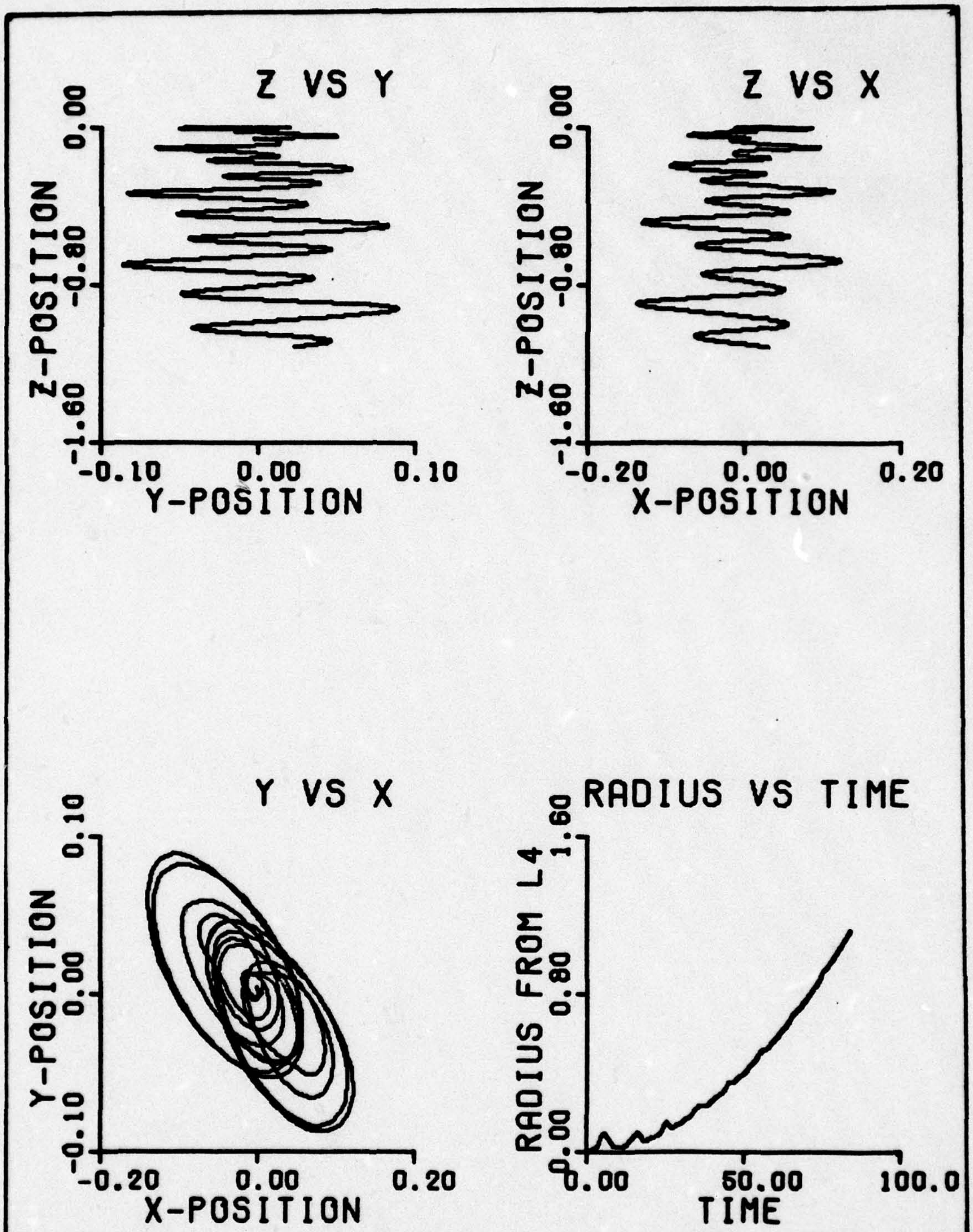


Figure 16. Trajectory with no Control (zero IC)

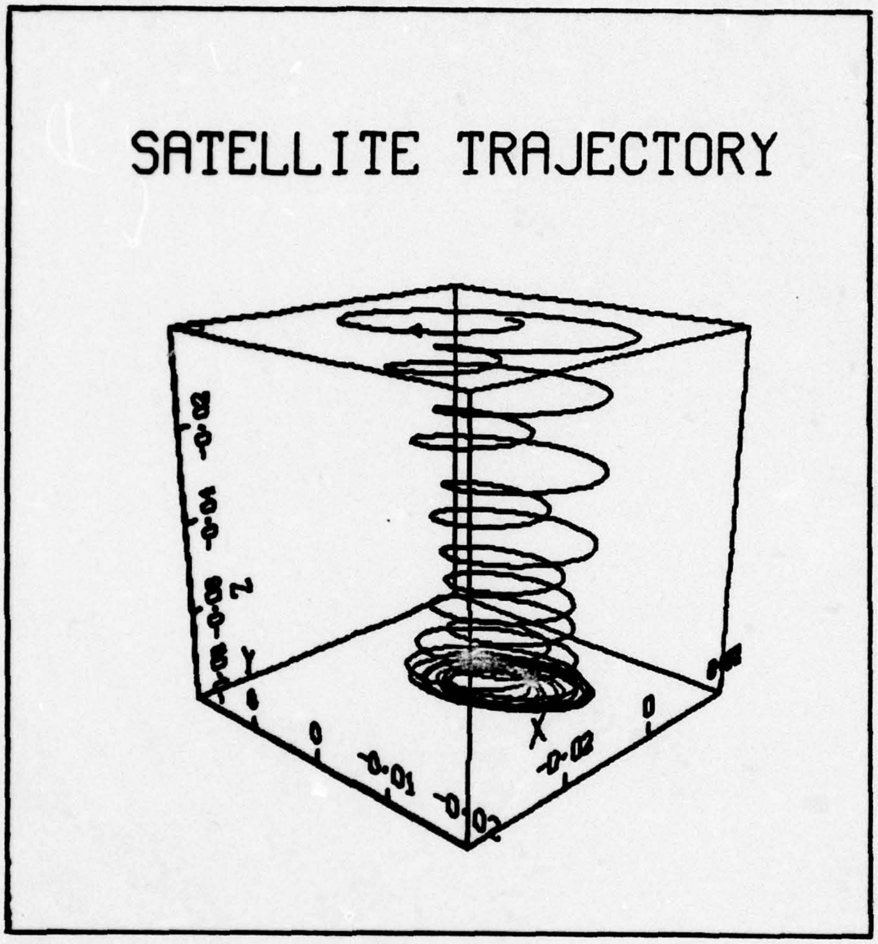


Figure 17

Satellite Trajectory

(q = .01; zero IC)

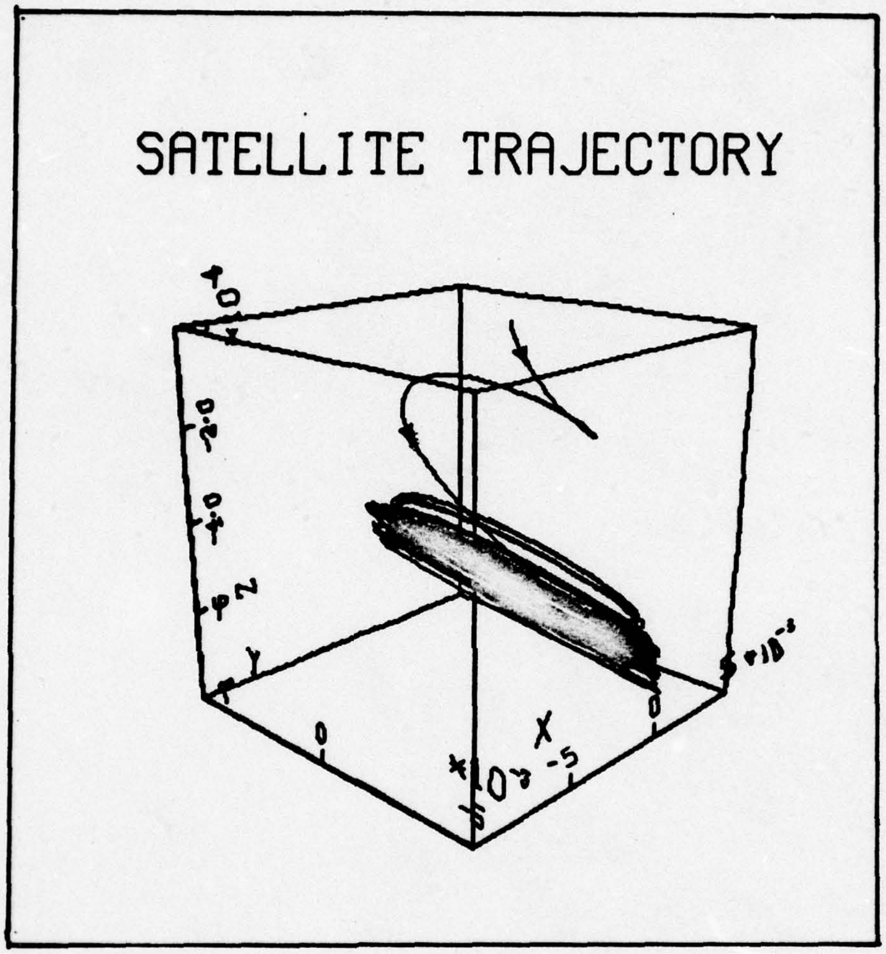


Figure 18

Satellite Trajectory

(q = 1.0; zero IC)

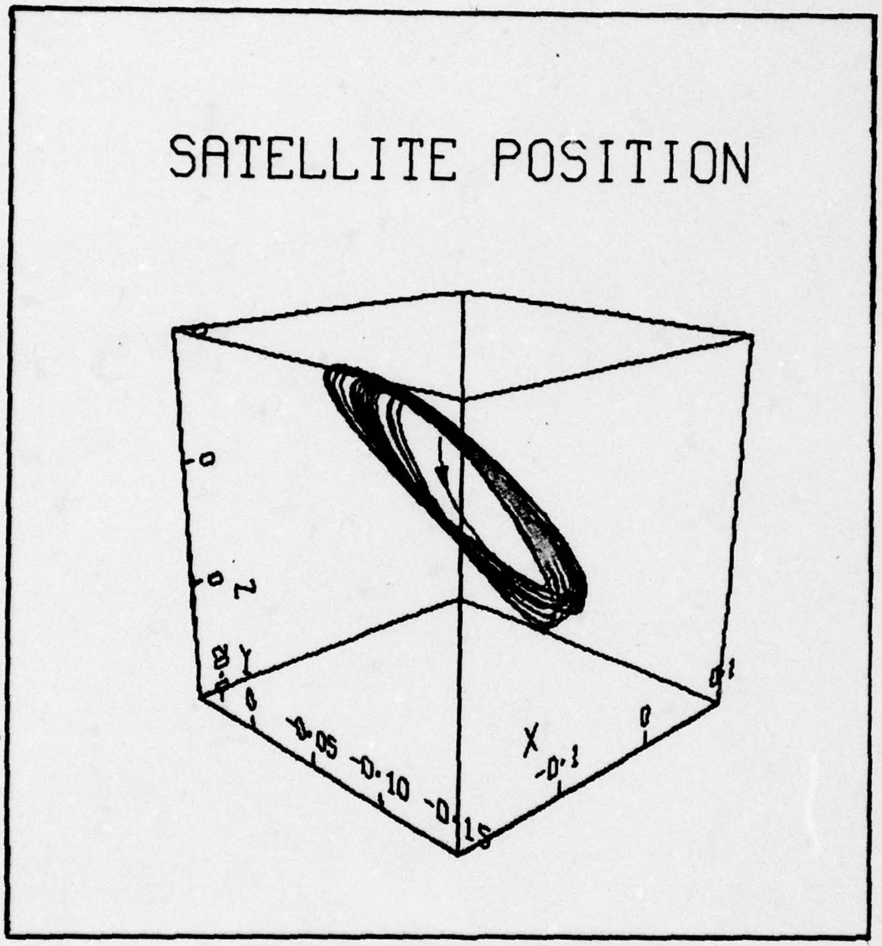


Figure 19

Satellite Trajectory

(q = 10.; zero IC)

ure 20 gives a typical plot of the velocity. It exhibits the same characteristics as the trajectory. As can be noted from equation (24) the control is formed by a combination of states, thus in the steady-state the control is periodic with a nearly constant magnitude.

The performance of the modified control can be summarized by Figure 21. The cost varies almost linearly from perfect control down to $q = .1$ and then asymptotically approaches zero as q is decreased. While perfect station keeping is the most costly option, controlling with q less than .1 is not desired because of the large increase in peak radius with minimal reduction in cost. Figure 21 will enable the user to choose the best option for his situation by evaluating cost versus tolerable distance from L4.

Effects of Initial Conditions

In chapter III it was shown that the gains vary with initial configuration of the earth, moon and sun. In this section, system performance will be studied in response to initial planet configurations as well as initial position and velocity conditions for the satellite. Figure 22 depicts the range of average thrust as α is varied while initial satellite position and velocity are zero. One can see that there are initial conditions that will minimize the cost for a given q. Figures 23 - 24 give an indication of why the cost varies. For the smaller α the transient interval is longer, i.e., the controller is scheduling a smaller thrust for a longer period of time. Thus, the cost is smaller. Note

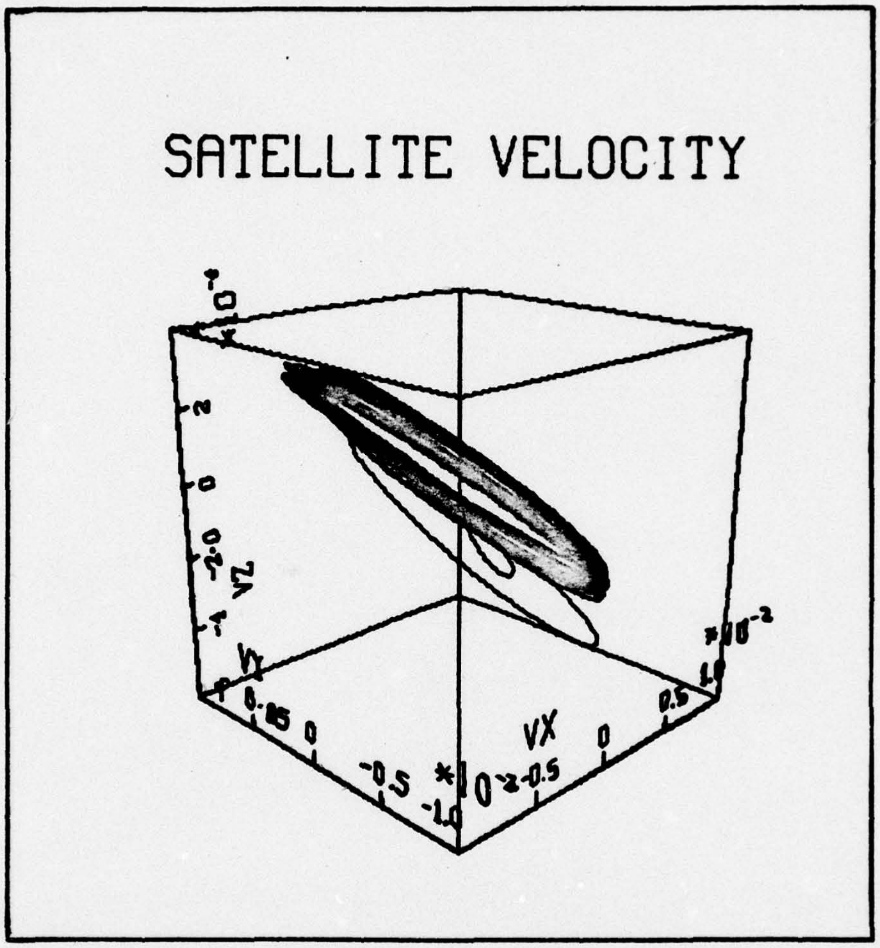


Figure 20

Typical Satellite Velocity

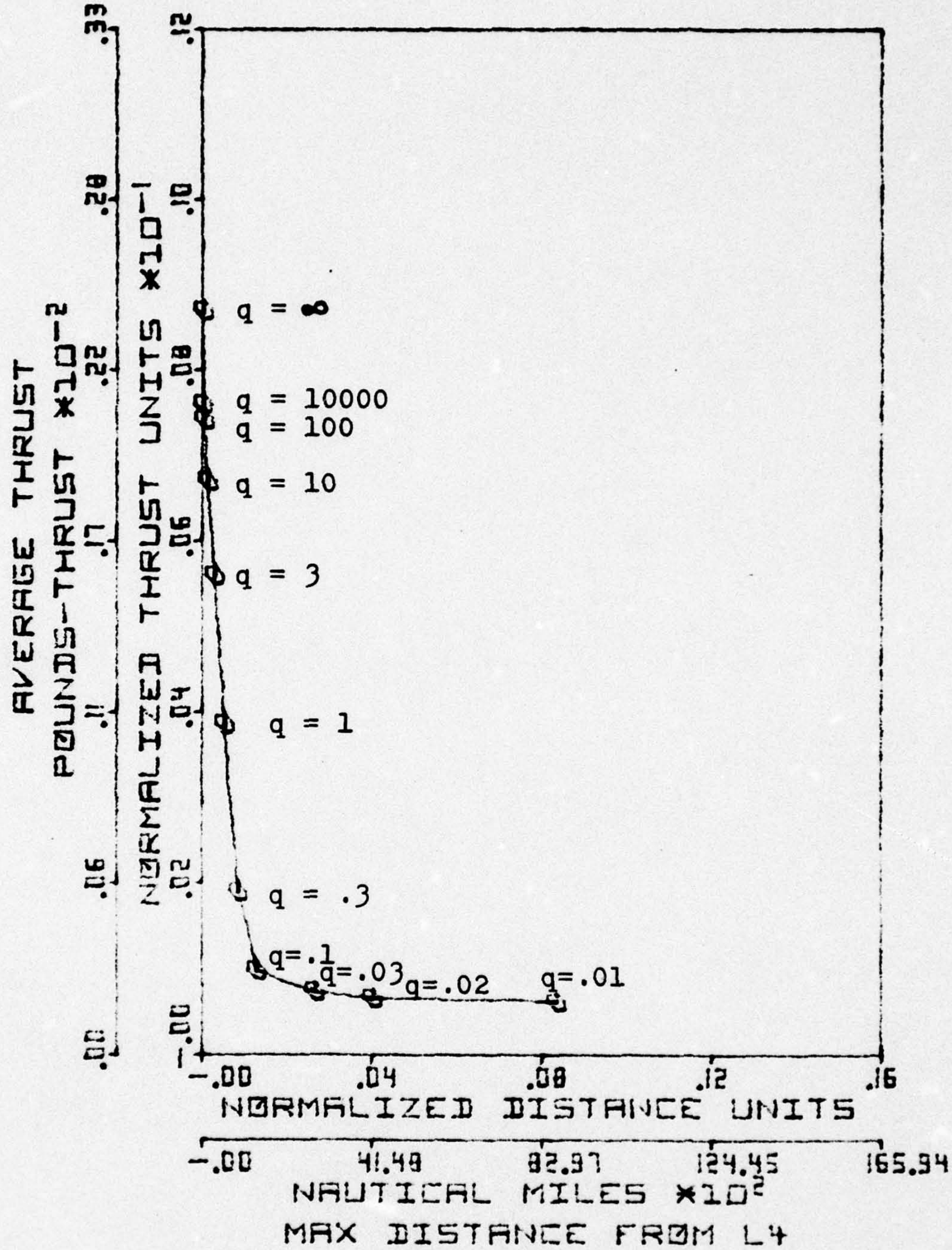


Figure 21

Total Cost VS Peak Drift

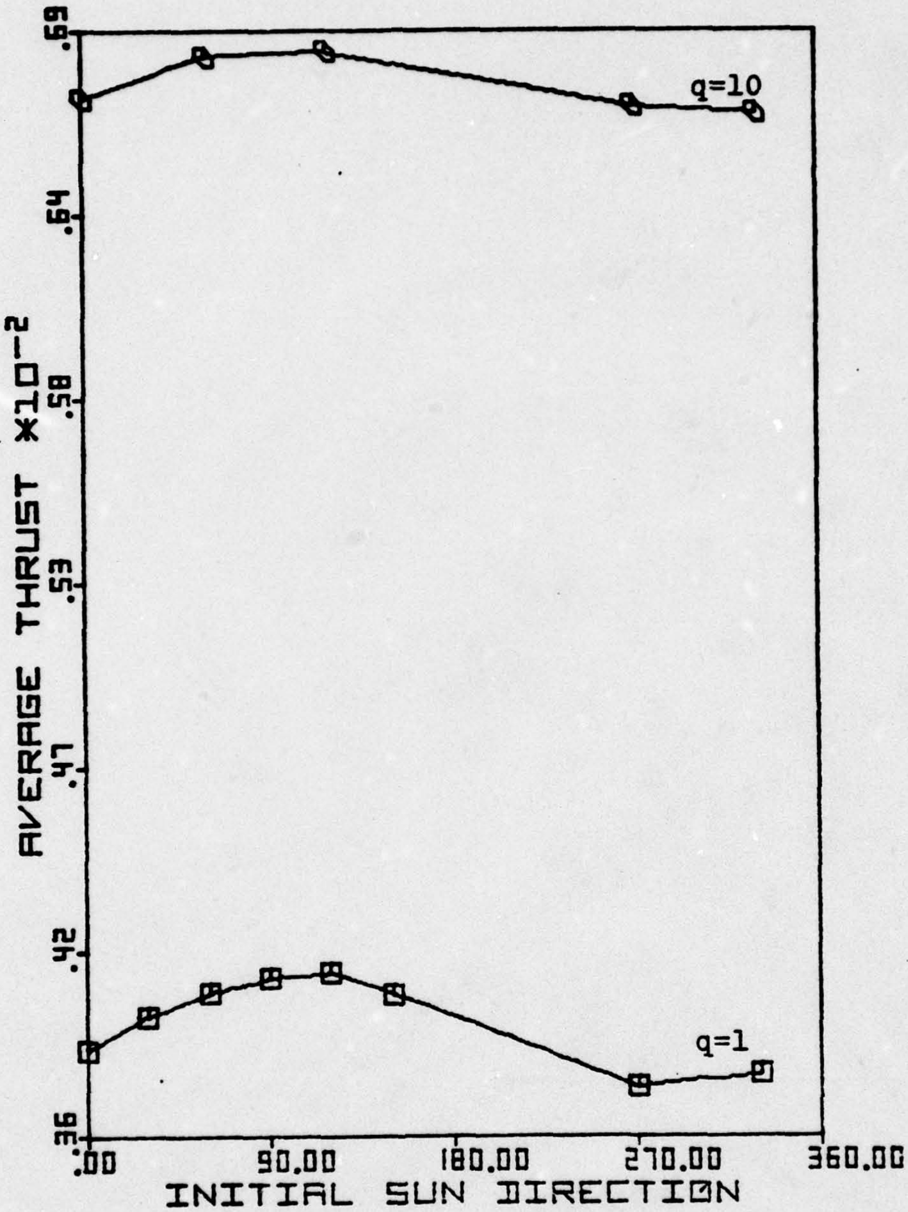


Figure 22

Average Thrust VS Initial Sun Direction

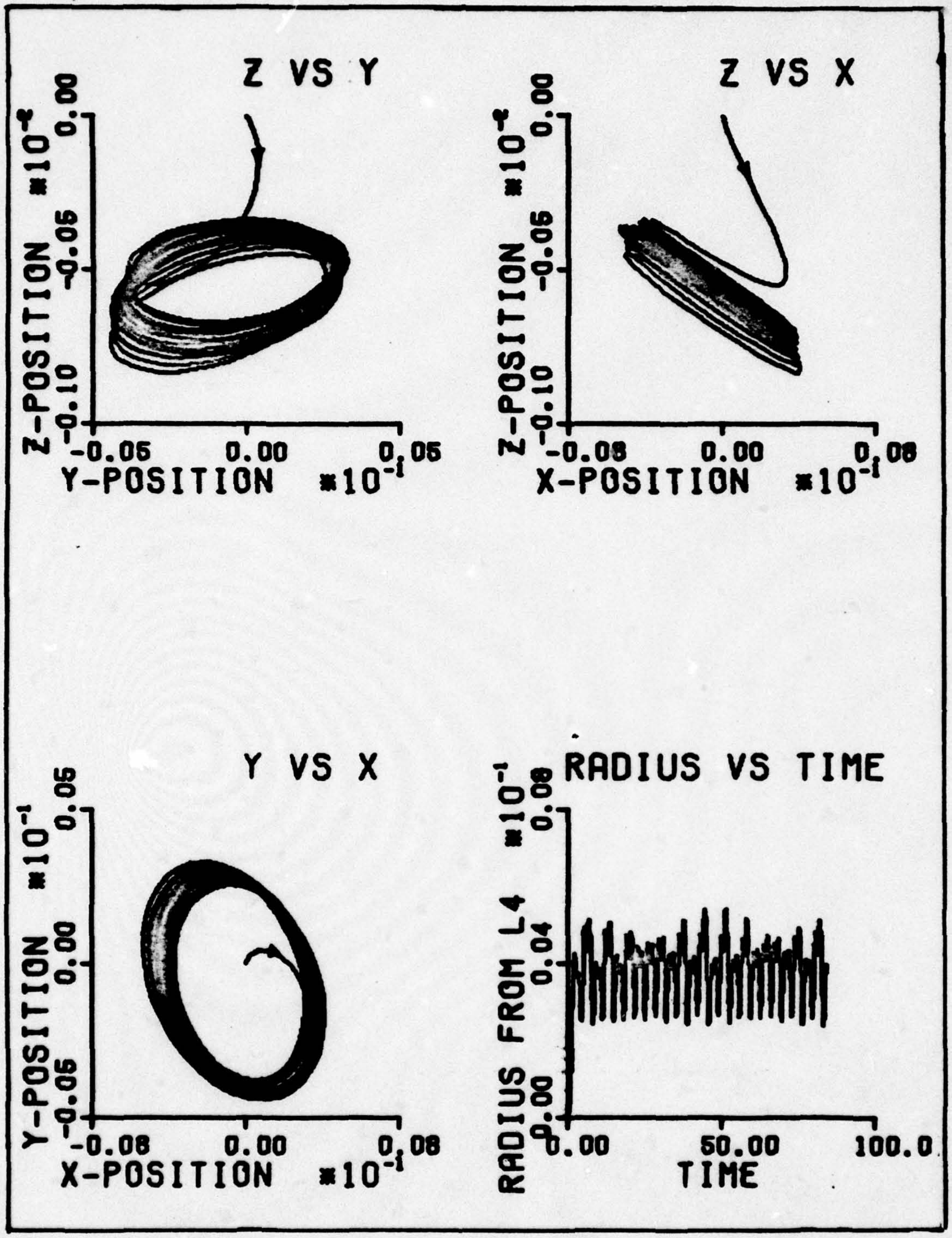


Figure 23

Satellite Trajectory

($q = 1$; $\alpha = 120$)

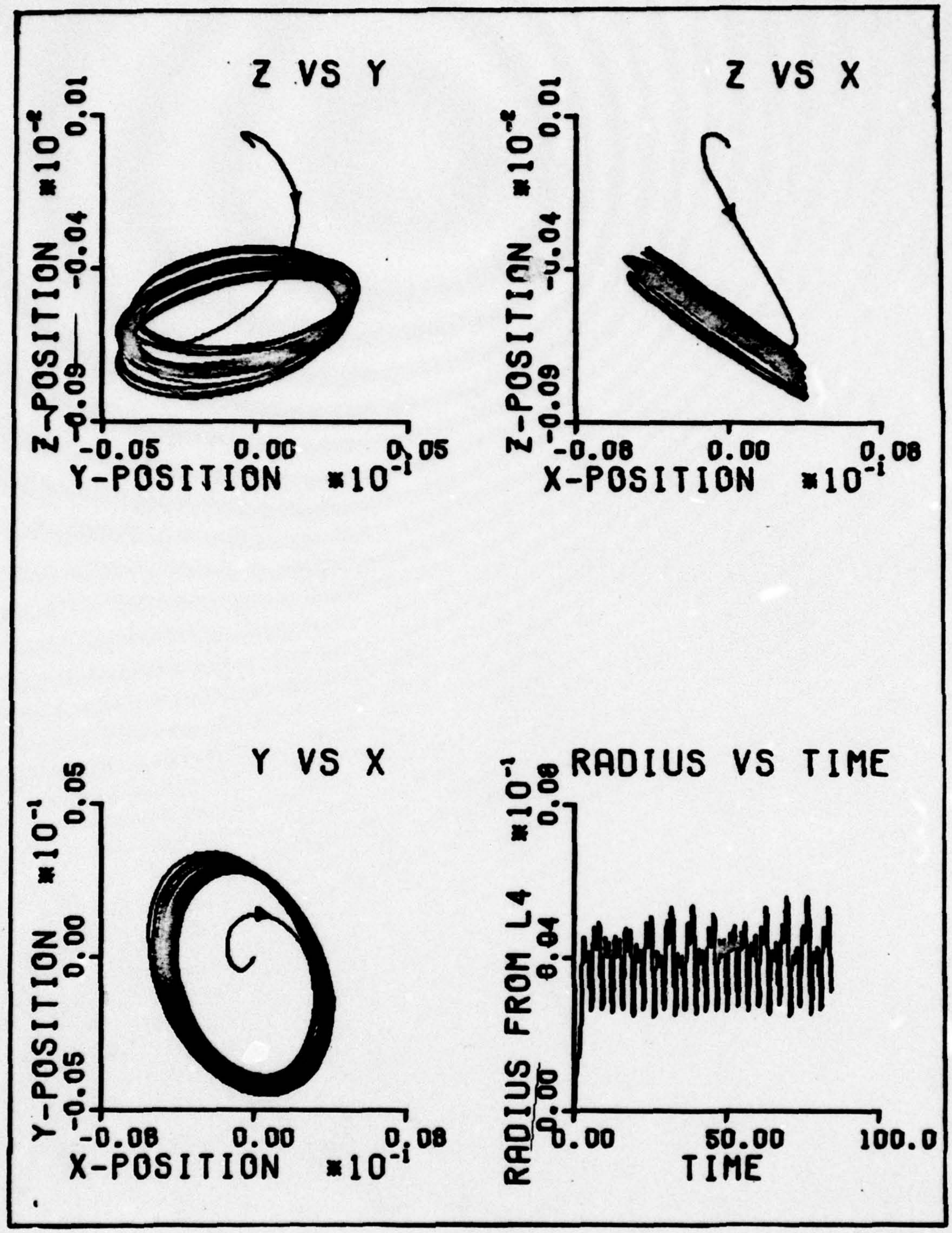


Figure 24

Satellite Trajectory ($q = 1$; $\alpha = 270^{o}$)

that for a given q, the steady-state path remains unchanged as α is varied. By proper selection of initial position and velocity, the transient phase can be eliminated; but, the total cost is higher. This is because the controller is initially scheduling higher thrust levels than it is in the corresponding transient time period. From Figure 22 one can also observe that as q is increased the dependance on initial conditions is decreased. This is a direct result of the shorter transient interval associated with the higher values of q. Note, that for $q = 1$, average thrust varies 9% and for $q = 10$ thrust varies 2.8% (Ref Figure 22).

V. Discussion

Model Deficiencies

The three-dimensional model used for the four body system is restricted by the following assumptions:

1. Only the gravitational perturbations from the sun, earth and moon were considered. Perturbative effects such as solar wind, gravitational pull from other planets, etc. were neglected.
2. All masses were considered point masses. The earth and sun were considered to be the centers of mass for their respective system.
3. The nonlinear equations were linearized about the L4 point.

Since the controller used is formed by a combination of the states, it is felt that model deficiencies were minor.

System Realizability

Thrust Requirements. The thrust requirements for station keeping are quite small. As an example, (Ref Fig. 21), a 1000-lb satellite would require .0017-lb thrust to confine its peak drift to a 500 nautical mile radius. So, for continuous, low thrust requirements an ion or monopropellant engine would be the most likely choice. Since the thrust vector is periodic, higher thrust engines which would fire a specified discrete times might provide a thrust which would approximate the continuous thrust provided by the modified controller.

Sensor Requirements. The control used requires continuous knowledge of the position and velocity states and the sun direction, ϕ. Laser ranging might be one method of providing the control system with this information. The sensor requirements should not be a serious problem. The constant feedback gains would have to be calculated *a priori* based on the desired launch date (defining initial solar system configuration) and the best estimate of the astronomical constants over the mission duration.

VI. Summary and Recommendations

Summary

The linearized equations of motion for a satellite stationed at or near the L4 lunar libration point were derived for the three-dimensional, four body case. A control system was then obtained by employing optimal control theory to this system of equations. Modification of the optimal controller using fixed gains resulted in a control system that was more computationally attractive while providing performance near the optimal.

Several observations can be made from the three-dimensional study of the controlled satellite. First, both trajectory and control become periodic after a brief transitory phase. This allows the definition of average thrust as the average magnitude of the control over one period. Second, initial conditions were shown to affect only the transient performance. Third, as q is increased the effects of initial solar configuration are reduced, i.e. smaller peak drift but longer thrust. Fourth, initial conditions which eliminate the transient phase result in a more costly trajectory than the trajectory with a transient phase. And finally, the range of station keeping costs is bounded. There is both an upper limit and a lower practical limit. This enables the user to make the tradeoff between maximum thrust he can afford and peak drift from the L4 point.

Recommendations

To finally validate the results of this study and previous ones, the control system should be tested using the nonlinear equations of motion. Using actual ephemeris data and comparing the results with this and previous studies will given an indication of the accuracy required for the astronomical constants.

Another area requiring further study is the effects of initial conditions. It would be useful to determine a method to predict the optimal initial conditions for a given weighting, q.

Additionally, a sensitivity analysis to determine the accuracy required in determination of the states by the sensors is necessary.

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Appendix A

Solar Perturbative Force

The solar perturbative force is given by (See glossary page vii for list of symbols used)

$$\delta \bar{F}_s = \frac{\bar{F}_s}{m_p} - \ddot{\bar{r}}_{sb} \tag{31}$$

but

$$\ddot{\bar{r}}_{sb} = \frac{\bar{F}_b}{M} = - \frac{k^2 \, M m_s}{M \, r_{sb}^3} \bar{r}_{sb}$$

where m_s is the mass of the sun and M is the mass of the earth and moon. Substituting this expression into equation (31), the force becomes

$$\delta \bar{F}_s = \frac{-k^2 \, M_s}{R_{sb}^2} \left[\frac{r_{sb}^3}{r_{sp}^3} \bar{r}_{sp} - \bar{r}_{sb} \right] \tag{32}$$

It can be shown that (Ref 7:84-89)

$$\left(\frac{r_{sb}}{r_{sp}}\right)^3 = \left(1 + \frac{2 \, \bar{r}_{sb} - \bar{r}}{r_{sb}^2} + \frac{r^2}{r_{sb}^2}\right)^{-3/2} \tag{33}$$

Using the expansion

$$(1 + z)^{-n} = 1 - nz + \frac{n(n + 1)}{2!} z^2 - \frac{n(n + 1) \, (n + 2)}{3!} z^3 + \dots$$

the right side of equation (33) becomes

$$\left(\frac{r_{sb}}{r_{sp}}\right)^3 = \left(\quad - \frac{3 \, \bar{r}_{sb} \cdot \bar{r}}{r_{sb}^2} - \frac{3 \, r^2}{2 \, r_{sb}^2} + \frac{15}{2} \frac{(\bar{r}_{sb} \cdot \bar{r})^2}{r_{sb}^4} + \frac{15}{8} \frac{r^4}{r_{sb}^4} + \dots \right) \tag{34}$$

Equation (31) then becomes

$$\delta\bar{F}_s = \frac{k^2 M_s}{r_{sb}^3} \{\bar{r}_{sb} - (1 - \frac{3\bar{r}_{sb}\cdot\bar{r}}{r_{sb}^2} - \frac{3r^2}{2r_{sb}^2} + \frac{15(\bar{r}_{sb}\cdot\bar{r})^2}{2\, r_{sb}^4} + \frac{15}{8}\frac{r^4}{r_{sb}^4}$$

$$+ \ldots)(\bar{r}_{sb} + r)\} \qquad (35)$$

Equation (35) is a vector equation with all vectors taken with respect to the rotating x,y,z-frame (see Figure 2). To transform the vectors from the I,J,K-frame to the rotating frame the following relations are used:

$$A = \begin{vmatrix} \text{Cos}\ \beta\ \text{Cos}\ (\theta + \phi) & \text{Cos}\ \beta\ \text{Sin}\ (\theta + \phi) & \text{Sin}\ \beta \\ -\ \text{Sin}\ (\theta + \phi) & +\ \text{Cos}\ (\theta + \phi) & 0 \\ -\ \text{Sin}\ \beta\ \text{Cos}\ (\theta + \phi) & -\ \text{Sin}\ \beta\ \text{Sin}\ (\theta + \phi) & \text{Cos}\ \beta \end{vmatrix}$$

$$\bar{r}_{sb}^{x} = A_I^x\, \bar{R}_{sb}^I = R_{sb} \begin{vmatrix} \text{Cos}\ \beta\ \text{Cos}\ \phi \\ -\ \text{Sin}\ \phi \\ -\ \text{Sin}\ \beta\ \text{Cos}\ \phi \end{vmatrix} ; \quad \bar{r} = \begin{vmatrix} \epsilon \\ \psi \\ \rho \end{vmatrix}$$

where

ϕ is the initial sun direction relative to the x,y,z-frame

β is the inclination of the ecliptic plane

To evaluate the vector relationships of Equation (35) the following are useful

$$\bar{r}_{sb} \cdot \bar{r} = R_{sb}\,(\epsilon \cos\beta \cos\phi - \psi \sin\phi - \rho \sin\beta \cos\phi)$$

$$(\bar{r}_{sb} \cdot \bar{r})\, \bar{r}_{sb} = R_{sb}^2 (\epsilon \cos\beta \cos\phi - \psi \sin\phi - \rho \sin\beta \cos\phi) \cdot \begin{vmatrix} \cos\beta\cos\phi \\ -\sin\phi \\ -\sin\beta\cos\phi \end{vmatrix}$$

$$(\bar{r}_{sb} \cdot \bar{r})\, \bar{r} = R_{sb} (\epsilon \cos\beta \cos\phi - \psi \sin\phi - \rho \sin\beta \cos\phi) \cdot \begin{vmatrix} \epsilon \\ \psi \\ \rho \end{vmatrix}$$

Using the above expressions, Eqn (35) is

$$\delta\bar{F}_s = N \left| - \begin{vmatrix} \epsilon \\ \psi \\ \rho \end{vmatrix} + 3(\epsilon \cos\beta \cos\phi - \psi \sin\phi - \rho \sin\beta \cos\phi) \cdot \begin{vmatrix} \cos\beta\cos\phi \\ -\sin\phi \\ -\sin\beta\cos\phi \end{vmatrix} \right.$$

$$+ \frac{3(\epsilon \cos\beta \cos\phi - \psi \sin\phi - \rho \sin\beta \cos\phi)}{R_{sb}} \begin{vmatrix} \epsilon \\ \psi \\ \rho \end{vmatrix}$$

$$+ \frac{3(\epsilon^2 + \psi^2 + \rho^2)}{2R_{sb}} \begin{vmatrix} \cos\beta\cos\phi \\ -\sin\phi \\ -\sin\beta\cos\phi \end{vmatrix}$$

$$\left. - \frac{15(\epsilon \cos\beta \cos\phi - \psi \sin\phi - \rho \sin\beta \cos\phi)^2}{2R_{sb}} \begin{vmatrix} \cos\beta\cos\phi \\ -\sin\phi \\ -\sin\beta\cos\phi \end{vmatrix} \right| \qquad (36)$$

In Eqn (36) terms with a factor smaller than $1/R_{sb}$ ($\approx .00^3$) in normalized time units) were neglected.

The equation for the solar perturbative force is written in terms of the earth-moon barycenter. To be compatible with the results of the three body portion, Eqn (36) must be expanded about the L4 point. This is accomplished by making the following definitions:

$$\epsilon = x_o + x; \ \psi = y_o + y; \ \rho = z$$

where (x_o, y_o, 0) are the coordinates of the L4 point in the x, y, z-frame. Expanding Eqn (36), the terms are

a) $$\begin{vmatrix} \epsilon \\ \psi \\ \rho \end{vmatrix} = - \begin{vmatrix} x_o + x \\ y_o + y \\ z \end{vmatrix}$$

b) $$\frac{3(\bar{r}_{sb} \cdot \bar{r})}{r_{sb}^2} \bar{r}_{sb} = \frac{3 R_{sb}^2 (\epsilon \cos \beta \cos \phi - \psi \sin \phi - \rho \sin \beta \cos \phi)}{R_{sb}^2} \begin{vmatrix} \cos \beta \cos \phi \\ - \sin \phi \\ - \sin \beta \cos \phi \end{vmatrix}$$

$$= 3 \begin{vmatrix} \tfrac{1}{4}(x_o + x)\,(1 + \cos 2\beta + \cos 2\phi + \cos 2\beta \cos 2\phi) - \tfrac{1}{2}(y_o + y) \cos \beta \sin 2\phi \\ - \tfrac{1}{4} z (\sin 2\beta + \sin 2\beta \cos 2\phi) \end{vmatrix}$$

$$\left|\begin{array}{l} -\tfrac{1}{2}(x_o + x)\cos\beta\sin 2\phi + \tfrac{1}{2}(y_o + y)(1 - \cos 2\phi) \\ \qquad\qquad + \tfrac{1}{2}z(\sin\beta\sin 2\phi) \\ -\tfrac{1}{4}(x_o + x)(\sin 2\beta + \sin 2\beta\cos 2\phi) + \tfrac{1}{2}(y_o + y)\sin\beta\sin 2\phi \\ \qquad\qquad + \tfrac{1}{4}z(1 - \cos 2\beta + \cos 2\phi - \cos 2\beta\cos 2\phi) \end{array}\right|$$

c) $$\frac{3(\bar{r}_{sb}\cdot\bar{r})}{r_{sb}^2} = \frac{3R_{sb}(\epsilon\cos\beta\cos\phi - \psi\sin\phi - \rho\sin\beta\cos\phi)}{R_{sb}^2} \left|\begin{array}{c} \epsilon \\ \psi \\ \rho \end{array}\right|$$

$$= \frac{3}{R_{sb}} \left|\begin{array}{l} (x_o^2 + 2x_o x)\cos\beta\cos\phi - (x_o y_o + x_o y + x y_o)\sin\phi - x_o z\sin\beta\cos\phi \\ (x_o y_o + x_o y + x y_o)\cos\beta\cos\phi - (y_o^2 + 2y_o y)\sin\phi - y_o z\sin\beta\cos\phi \\ (x_o z)\cos\beta\cos\phi \qquad - y_o z\sin\phi \end{array}\right|$$

d) $$\frac{3r^2}{2r_{sb}^2}\bar{r}_{sb} = \frac{3(\epsilon^2 + \psi^2 + \rho^2)R_{sb}}{2R_{sb}^2} \left|\begin{array}{l} \cos\beta\cos\phi \\ -\sin\phi \\ -\sin\beta\cos\phi \end{array}\right|$$

$$= \frac{3}{2R_{sb}} \left|\begin{array}{l} (x_o^2 + 2x_o x)\cos\beta\cos\phi + (y_o^2 + 2y_o y)\cos\beta\cos\phi \\ -(x_o^2 + 2x_o x)\sin\phi - (y_o^2 + 2y_o y)\sin\phi \\ -(x_o^2 + 2x_o x)\sin\beta\cos\phi - (y_o^2 + 2y_o y)\sin\beta\cos\phi \end{array}\right|$$

e) $$-\frac{15}{2}\,\frac{(\bar{r}_{sb}\cdot\bar{r})^2}{r_{sb}^4}\,\bar{r}_{sb} = -\frac{15\,R_{sb}^3\,(\epsilon\cos\beta\cos\phi - \psi\sin\phi - \rho\sin\beta\cos\phi)}{2\,R_{sb}^4}\left|\begin{matrix} c\beta\ c\phi \\ -\ s\phi \\ -\ s\beta\ c\phi \end{matrix}\right|$$

$$= -\frac{15}{2R_{sb}}\left|\begin{matrix} \frac{1}{16}(x_o + x)^2\ (9\ c\beta\ c\phi + 3\ c\beta\ c3\phi + 3\ c3\beta\ c\phi + c3\beta\ c3\phi) \\ -\frac{1}{4}(x_o + x)^2\ (s\phi + c2\beta\ s\phi + s\phi\ c2\phi + c2\beta\ s\phi\ c2\phi) \\ -\frac{1}{8}(x_o + x)^2\ (3s\beta\ c\phi + s\beta\ c3\phi + 3\ s\beta\ c2\beta\ c\phi + s\beta\ c2\beta\ c3\phi) \end{matrix}\right|$$

$$+\left|\begin{matrix} -\frac{1}{2}(x_oy_o + x_oy + xy_o)\ (s\phi + c2\beta\ s\phi + c2\phi\ s\phi + c2\beta\ c2\phi\ s\phi \\ (x_oy_o + x_oy + xy_o)\ (c\beta\ s\phi - c\beta\ c\phi\ c2\phi) \\ \frac{1}{2}(x_oy_o + x_oy + xy_o)\ (s2\beta\ s\phi + s\phi\ s2\beta\ c2\phi) \end{matrix}\right|$$

$$+\left|\begin{matrix} \frac{1}{2}(y_o^2 + 2y_oy)\ (c\beta\ c\phi - c\beta\ c\phi\ c2\phi) \\ -\frac{1}{4}(y_o^2 + 2y_oy)\ (3\ s\phi - s3\phi) \\ -\frac{1}{2}(y_o^2 + 2y_oy)\ (s\beta\ c\phi - s\beta\ c\phi\ c2\phi) \end{matrix}\right|$$

e) (con't)

$$+ \left| \begin{array}{l} -\frac{1}{4} (x_o z) (3 \; s\beta \; c\phi + s\beta \; c3\phi + 3s\beta \; c2\beta \; c\phi + s\beta \; c2\beta \; c3\phi) \\ \frac{1}{2} (x_o z) (\sin 2\beta \sin \phi + \sin 2\beta \sin \phi \cos 2\phi) \\ \frac{1}{4} (x_o z) (3 \; c\beta \; c\phi - 3 \; C\beta \; c2\beta \; c\phi + c\beta \; c3\phi - c\beta \; c2\beta \; c3\phi) \end{array} \right|$$

$$+ \left| \begin{array}{l} \frac{1}{2} y_o z \; (s2\beta \; s\phi + s2\beta \; s\phi \; c2\phi) \\ - y_o z \; (s\beta \; c\phi - s\beta \; c\phi \; c2\phi) \\ -\frac{1}{2} y_o z \; (s\phi + s\phi \; c2\phi - c2\beta \; s\phi - c2\beta \; s\phi \; c2\phi) \end{array} \right|$$

In the above expansions, the higher order perturbation terms (x^2, y^2, z^2, xy, xz, yz) will be much smaller than the first order perturbation terms and have been neglected. Table VI represents a summary of the expressions a) -e) with like terms grouped.

TABLE VI

Expansion and Grouping of Terms Eq (36)

Term	Expansion in terms of double and triple angle and consolidation of like terms
a)	$- \begin{vmatrix} x_o + x \\ y_o + y \\ z \end{vmatrix}$
b)	$3 \begin{vmatrix} \frac{1}{4}(x_o + x)(1 + c2\beta + c2\emptyset + c2\beta\ c2\emptyset) & -\frac{1}{2}(y_o + \dot{y})\ c\beta\ s2\emptyset & -\frac{z}{4}(s2\beta + s2\beta\ c2\emptyset \\ -\frac{1}{2}(x_o + x)(c\beta\ s2\emptyset) & +\frac{1}{2}(y_o + y)(1 - c2\emptyset) & +\frac{z}{4}\ s\beta\ s2\emptyset \\ -\frac{1}{4}(x_o + x)(s2\beta + s2\beta\ c2\emptyset) & +\frac{1}{2}(y_o + y)(s2\beta\ s2\emptyset) & +\frac{z}{4}(1 - c2\beta + c2\emptyset \\ & & - c2\beta\ c2\emptyset) \end{vmatrix}$
x_o^2, y_o^2 of c,d,e	$\frac{1}{R_{sb}} \begin{vmatrix} (\frac{9}{32} c\beta\ c\emptyset - \frac{45}{32}(3B\ c\emptyset) x_o^2 & -(\frac{9}{4}\cos\beta\cos\emptyset) y_o^2 \\ (\frac{3}{8} s\emptyset + \frac{15}{8} s\emptyset \cos 2\beta) x_o^2 & + \frac{9}{8} y_o^2 \sin\emptyset \\ (\frac{21}{16} x_o^2 + \frac{9}{4} y_o^2) \sin\beta\cos\emptyset & \end{vmatrix}$

TABLE VI (con't)

Term	Expansion in terms of double and triple angle and consolidation of like terms
$x_o y$, $y y_o$ $x y_o$, $x x_o$ of c,d,e	$\frac{9x}{8R_{sb}} \left\| \begin{array}{ll} \frac{1}{2}x_o\ (c\beta\ c\phi\ -5c3\beta\ c\phi) & + \frac{2}{3}y_o(s\phi + 5s\phi\ c2\beta) \\ \frac{2}{3}x_o\ (s\phi + 5s\phi\ c2\beta) & - 4y_o\ c\beta\ c\phi \\ \frac{7}{3}x_o\ s\beta c\phi & - \frac{10}{3}y_o\ s2\beta\ s\phi \end{array} \right\|$
	$+ \frac{9y}{8R_{sb}} \left\| \begin{array}{ll} \frac{2}{3}x_o\ (s\phi + 5s\phi\ c2\beta) & - 4y_o\ c\beta\ c\phi \\ -4x_o\ c\beta\ c\phi & + 2y_o\ s\phi \\ - \frac{10}{3}x_o\ s2\beta\ s\phi + 4y_o\ s\beta\ c\phi & \end{array} \right\|$
$x_o z$, $y_o z$ from c, d,e	$\frac{9z}{8R_{sb}} \left\| \begin{array}{l} \frac{7}{3}x_o\ s\beta\ c\phi - \frac{10}{3}y_o\ \sin 2\beta\ \sin\phi \\ -\frac{10}{3}x_o\ \sin 2\beta\ \sin\phi + 4y_o\ s\beta\ c\phi \\ -\frac{7}{3}x_o\ c\beta\ c\phi + \frac{2}{3}y_o\ \sin\phi \end{array} \right\|$
$x_o y_o$ from c, d,e	$\frac{x_o y_o}{R_{sb}} \left\| \begin{array}{l} \frac{3}{4}\ s\phi + \frac{15}{4}\ s\phi\ c2\beta \\ - \frac{9}{2}\ c\beta\ c\phi \\ - \frac{15}{4}\ s2\beta\ s\phi \end{array} \right\|$

Letting $n = m_s/R_{sb}^3$ and with $k^2 = 1$, Eqn (37) is

$$\dot{\delta}\overline{F}_s\, N = \left| - \begin{vmatrix} x_o + x \\ y_o + y \\ z \end{vmatrix} + \begin{vmatrix} \frac{3}{4}(x_o + x)\,(1 + \cos 2\beta) \\ \frac{3}{2}(y_o + y) \\ -\frac{3}{4}(x_o + x)\sin 2\beta \end{vmatrix} + 3 \begin{vmatrix} -\frac{z}{4}\sin 2\beta \\ 0 \\ \frac{z}{4}(1 - \cos 2\beta) \end{vmatrix} \right|$$

$$+ 3\cos 2\phi \begin{vmatrix} \frac{(x_o + x)}{9}(1 + \cos 2\beta) \\ -(\frac{y_o + y}{2}) \\ -(\frac{x_0 + x}{4})\sin 2\beta \end{vmatrix} + 3\cos 2\phi \begin{vmatrix} -\frac{z}{4}\sin 2\beta \\ 0 \\ \frac{z}{4}(1 - \cos 2\beta) \end{vmatrix}$$

$$+ 3\sin 2\phi \begin{vmatrix} -\frac{1}{2}(y_o + y)\cos\beta \\ -\frac{1}{2}(x_o + x)\cos\beta + \frac{z}{4}\sin\beta \\ \frac{1}{2}(y_o + y)\sin\beta \end{vmatrix} + \frac{9}{8R_{sb}} \begin{vmatrix} (\frac{1}{4}x_o^2 - 2y_o^2)\, c\beta\, c\phi - \frac{5}{4}x_o^2\, c3\beta\, c\phi \\ (\frac{1}{3}x_o^2 + y_o^2)\, s\phi + \frac{5}{3}x_o^2\, c2\beta\, s\phi \\ (\frac{7}{6}x_o^2 + 2y_o^2)\, s\beta\, c\phi \end{vmatrix}$$

$$+ \frac{9\,x}{8R_{sb}} \begin{vmatrix} \frac{1}{2}x_o(c\beta\, c\phi - 5\, c3\beta\, c\phi) + \frac{2}{3}y_o\,(s\phi + 5\, s\phi\, c2\beta) \\ \frac{2}{3}x_o\,(s\phi + 5\, s\phi\, c2\beta) - 4y_o\, c\beta\, c\phi \\ \frac{7}{3}x_o\, s\beta\, c\phi - \frac{10}{3}y_o\, s2\beta\, s\phi \end{vmatrix}$$

$$+ \frac{9y}{8R_{sb}} \left| \begin{array}{l} \frac{2}{3}x_o \ (s\phi + 5\ s\phi\ c2\beta) - 4y_o\ c\beta\ c\phi \\ -4x_o\ c\beta\ c\phi + 2y_o\ s\phi \\ -\frac{10}{3}x_o\ s2\beta\ s\phi + 4y_o\ s\beta\ c\phi \end{array} \right|$$

$$+ \frac{9z}{8R_{sb}} \left| \begin{array}{l} \frac{7}{3}x_o\ s\beta\ c\phi - \frac{10}{3}y_o\ s2\beta\ s\phi \\ -\frac{10}{3}x_o\ s2\beta\ s\phi + 4y_o\ s\beta\ c\phi \\ -\frac{7}{3}x_o\ c\beta\ c\phi + \frac{2}{3}y_o\ s\phi \end{array} \right|$$

$$+ \frac{x_o\ y_o}{R_{sb}} \left| \begin{array}{l} \frac{3}{4}\ s\phi + \frac{15}{4}\ s\phi\ c2\beta \\ -\frac{9}{2}\ c\beta\ c\phi \\ -\frac{15}{4}\ s2\beta\ s\phi \end{array} \right| \Bigg|$$

Appendix B

FORTRAN Program for Gain Computations

The program listed in this appendix is used for the time varying gain computations. To use the program, the user must attach and library the IMSL package, CC6600. This provides access to the integration routine, ODE.

The output is both printed and on tape. Tape 6 contains the time varying gain schedule used for the optimal performance and used as input to compute the constant gain.

The input data is listed and explained in the program listing.

```
      PROGRAM GAIN(INPUT,OUTPUT,TAPE5,TAPE6)
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                     C
C     THIS PROGRAM CALCULATES THE VARYING GAINS USED TO CONTOL A      C
C     SATELLITE AT THE L4 POINT.                                      C
C     INPUT DATA:                                                     C
C          E-     ECCENTRICITY OF THE LUNAR ORBIT                     C
C          E1-    ECCENTRICITY OF THE EARTH ORBIT                     C
C          BETA-  INCLINATION OF THE ECLIPTIC PLANE                   C
C          OM-    DIFFERENTIAL RADIAN FREQUENCY                       C
C          ALF-   THE INITIAL SUN DIRECTION RELATIVE TO THE I,J,K- FRAMEC
C          Q-     WEIGHT ON THE FINAL STATES                          C
C          MG-    FINAL TIME FOR THE INTEGRATION                      C
C     OUTPUT OF THIS PROGRAM IS:                                      C
C          1.  PRINTOUT OF THE VARYING GAINS                          C
C          2.  TAPE6- DATA FOR INPUT TO THE COST PROGRAM              C
C     SUBROUTINES USED:                                               C
C          1. ODE (CC6600 LIBRARY)-INTEGRATES THE RICCATI EQUATION    C
C          2. FA- USED BY ODE TO COMPUTE THE VALUE OF DK AT EACH TIME C
C          3. CONST- COMPUTES THE VALUES FOR THE CONSTANTS OF         C
C              THE A MATRIX AT EACH TIME, DT                          C
C          4. EANOM- COMPUTES THE ECCENTRIC ANOMALY GIVEN THE         C
C              MEAN MOTION AND THE ECCENTRICITY                       C
C                                                                     C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      COMMON/COM1/A(9,9),Q,GG(9,9),AT,OM,ALF,GAM
      DIMENSION Z(45),X(9),C(21)
      DIMENSION WK(1200), IWK(5)
      EXTERNAL FA,EANOM
      NAMELIST/N1/E,E1,BETA,ISP,OM,GAM,ALF
      NAMELIST/N4/Q,DQ,MG
      READ N1
      READ N4
      PRINT*
      PRINT*,"     INPUT DATA:  E= ",E,"     E1= ",E1,"     BETA= ",BETA
```

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```
      PRINT*,"                              GAM= ",GAM,"      ALF= ",ALF,"      Q= ",Q
      PRINT* $PRINT*
      PRINT 151
151   FORMAT(50X,"***** KALMAN GAIN ELEMENTS *****",//)
      PRINT 15,ISP,ISP,ISP,ISP,ISP,ISP,ISP,ISP,ISP
15    FORMAT(11X,4HTIME,4X,2HK(,I1,3H,1),7X,2HK(,I1,3H,2),7X,2HK(,I1,
     +3H,3),7X,2HK(,I1,3H,4),7X,2HK(,I1,3H,5),7X,2HK(,I1,3H,6),7X,
     +2HK(,I1,3H,7),7X,2HK(,I1,3H,8),7X,2HK(,I1,3H,9)/)
      DT=-.1
      DO 6 I=1,9
      DO 6 J=1,9
      A(I,J)=0.
6     GG(I,J)=0.
      DO 12 I=1,45
12    Z(I)=0.
      T=MG
      NF=T/ABS(DT)+1.1
      NA=1
      WRITE(5,*) NF
      TOUT=T+DT
      RE=AE=1.E-7
      IFLAG=1
      K=0
      DO 199 I=1,3
      K=K+2
      IF(K.EQ.6) K=7
      WRITE (6,14) (GG(K,J),J=1,9)
      WRITE(5,17) T,(GG(K,J),J=1,9)
      PRINT 13,NA,T,(GG(K,J),J=1,9)
199   CONTINUE
7     CONTINUE
      ALFD=ALF
      DO 53 NA=2,NF
      ALF=3.1415927*ALFD/180.-.0040216*T
      CALL CONST(C,E,E1,BETA,T)
```

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```
      DM1=2.*(OM*T+ALF)
      A(1,2)=1.
      A(2,1)=C(2)+C(3)*COS(DM1)
      A(2,3)=C(4)-C(5)*SIN(DM1)
      A(2,4)=C(1)
      A(2,5)=C(6)
      A(2,6)=-C(15)*(1+COS(DM1))
      A(2,8)=C(7)
      A(2,9)=-C(8)
      A(3,4)=1.
      A(4,1)=C(9)-C(5)*SIN(DM1)
      A(4,2)=-C(1)
      A(4,3)=C(10)-C(11)*COS(DM1)
      A(4,5)=C(12)
      A(4,6)=C(19)*SIN(DM1)
      A(4,8)=-C(13)
      A(4,9)=-C(14)
      A(6,7)=1.
      A(7,1)=-C(15)*(1+COS(DM1))
      A(7,3)=C(16)*SIN(DM1)
      A(7,5)=-C(17)
      A(7,6)=C(20)+C(21)*COS(DM1)
      A(7,8)=-C(17)
      A(7,9)=C(18)
      A(8,9)=-2.*OM
      A(9,8)=2.*OM
      CALL ODE(FA,45,Z,T,TOUT,RE,AE,IFLAG,WK,IWK)
      TOUT=T+DT
      K=1
      DO 16 L=1,7
      DO 16 J=L,9
      GG(L,J)=Z(K)
      GG(J,L)=Z(K)
16    K=K+1
17    FORMAT(0PF7.2,1P9E13.4)
```

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```
      K=0
      DO 198 I=1,3
      K=K+2
      IF(K.EQ.6) K=7
      WRITE(6,14)  (GG(K,J),J=1,9)
      WRITE(5,17) T,(GG(K,J),J=1,9)
      PRINT 13,NA,T,(GG(K,J),J=1,9)
14    FORMAT(1P9E13.4)
198   CONTINUE
53    CONTINUE
13    FORMAT(I7,0PF8.2,1P9E13.4)
      STOP "     GRACEFULLY"
      END
```

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                     C
C       ****** SUBROUTINE FA ******                                   C
C                                                                     C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      SUBROUTINE FA(T,Z,DZ)
      COMMON/COM1/A(9,9),Q,GG(9,9),AT,OM,ALF
      DIMENSION Z(45),DZ(45),ZZ(9,9),ZA(9,9),ATZ(9,9),ZBZ(9,9)
      K=1
      DO 7 I=1,9
      DO 7 J=I,9
      ZZ(I,J)=Z(K)
      ZZ(J,I)=Z(K)
7     K=K+1
      DO 3 I=1,9
      DO 3 J=I,9
      ZA(I,J)=0.
      ATZ(I,J)=0.
      ZBZ(I,J)=ZZ(I,2)*ZZ(2,J)+ZZ(I,4)*ZZ(4,J)+ZZ(I,7)*ZZ(7,J)
      DO 3 K=1,9
      ZA(I,J)=ZZ(I,K)*A(K,J)+ZA(I,J)
3     ATZ(I,J)=A(K,I)*ZZ(K,J)+ATZ(I,J)
      K=1
      DO 11 I=1,9
      DO 11 J=I,9
      DZ(K)=-ZA(I,J)-ATZ(I,J)+ZBZ(I,J)
11    K=K+1
      DZ(1)=DZ(1)-Q
      DZ(18)=DZ(18)-Q
      DZ(36)=DZ(36)-Q
      RETURN
      END
```

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                       C
C      ****** SUBROUTINE CONST ******                                   C
C                                                                       C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      SUBROUTINE CONST(C,E,E1,BETA,T)
      COMMON/COM1/A(9,9),Q,GG(9,9),AT,OM,ALF,GAM
      DIMENSION C(21)
      REAL NO
      OMEGA=.07480133
      ASUN=1.4959965E8
      AMASS=1/82.301
      AMSUN=328905.2
      AMOON=384400
      ASNORM=ASUN/(AMOON*(1-E))
      AM1=GAM*T
      CALL EANOM(AM1,E,UM)
      AM2=ALF+OMEGA*T
      CALL EANOM(AM2,E1,US)
      RS=ASNORM*(1-E1*COS(US))
      SUN=AMSUN/RS**3
      FM=1/(1-E*COS(UM))
      QM=1-2.*AMASS
      NO=SQRT(3.)*(1-E*COS(UM))/(2.*(1-E))
      EO=QM*NO/SQRT(3.)
      ESQ=SQRT(1-E**2)
      EE=1-E**2
      C(1)=2.*ESQ*FM**2
      C(2)=EE*FM**4-.25*FM**3-SUN*(1-3.*COS(2.*BETA))/4
      C(3)=.75*SUN*(1.+COS(2.*BETA))
      C(4)=1.29904*QM*FM**3-2.*E*ESQ*SIN(UM)*FM**4
      C(5)=1.5*SUN*COS(BETA)
      C(6)=SUN*(3.*COS(2.*BETA)-1)*EO/4
      C(7)=.75*SUN*(1.+COS(2.*BETA))*EO
      C(8)=1.5*SUN*COS(BETA)*NO
```

```
C(9)=1.29904*QM*FM**3+2.*E*ESQ*SIN(UM)*FM**4
C(10)=1.25*FM**3+EE*FM**4+0.5*SUN
C(11)=1.5*SUN
C(12)=.5*SUN*NO
C(13)=C(11)*NO
C(14)=C(5)*EO
C(15)=.75*SUN*SIN(2.*BETA)
C(16)=C(11)*SIN(BETA)
C(17)=C(15)*EO
C(18)=C(13)*SIN(BETA)
C(19)=.75*SUN*SIN(BETA)
C(20)=-FM**3-SUN*(1+3.*COS(2.*BETA))/4
C(21)=.75*SUN*(1-COS(2.*BETA))
RETURN
END
```

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                      C
C     ****** SUBROUTINE FANOM ******                                   C
C                                                                      C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      SUBROUTINE EANOM(AM,E,U)
      U0=AM+(E-(E**3)/8.+(E**5)/192.)*SIN(AM)+((E**2)/2.-(E**4)/6.+(E**6
     +)/48.)*SIN(2.*AM)+((3.*E**3)/8.-(27.*E**5)/128.)*SIN(3.*AM)+(
     +(E**4)/3.-(4.*E**6)/15.)*SIN(4.*AM)+(125.*E**5*SIN(5.*AM))/384.
 105  AM0=U0-E*SIN(U0)
      DELU=(AM-AM0)/(1-E*SIN(U0))
      IF(DELU.LT.1.E-8) GO TO 106
      U0=U0+DELU
      GO TO 105
 106  U=U0+DELU
      RETURN
      END
```

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```
      PROGRAM INVERT(INPUT,OUTPUT,TAPE5,TAPE6,TAPE7)
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                       C
C     THIS PROGRAM IS USED TO PREPARE DATA FOR INPUT TO THE             C
C     PERFORMANCE PROGRAM, "COST",  INVERT HAS TWO PRIMARY FUNCTIONS:   C
C           1.  OPTIMUM GAIN SCHEDULE GENERATED BY "GAIN" IS REVERSED   C
C               SO THAT THE GAIN ELEMENTS RANGE FROM TO TO TF.          C
C           2.  TIME VARYING GAINS ARE AVERAGED OVER STEADY STATE       C
C               TO YIELD THE CONSTANT GAIN MATRIX OF "TAPE7".           C
C     INPUT DATA:                                                       C
C           TAPE6 GENERATED BY THE GAIN COMPUTATION PROGRAM, "GAIN".    C
C     OUTPUT:                                                           C
C           1.  TAPE5 - USED FOR VARYING GAIN INPUT TO "COST"           C
C           2.  TAPE7 - USED FOR FIXED GAIN INPUT TO "COST"             C
C           3.  PRINTOUT OF CONSTANT GAINS                              C
C                                                                       C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      DIMENSION GR(3,9,505)
      DIMENSION XMIN(3,9,20),XMAX(3,9,20),IC1(3,9),IC2(3,9)
      PRINT*,"    ENTER N < NO. OF DATA ITEMS TO BE CHANGED>: "
      READ*,N
      DO 1 NA=1,N
      DO 1 K=1,3
      NB=N+1-NA
 1    READ (6,151) (GR(K,J,NB),J=1,9)
      DO 3 NA=1,N
      DO 3 K=1,3
 3    WRITE (5,151) (GR(K,J,NA),J=1,9)
 151  FORMAT(1P9E13.4)
      N=N-100
      DO 7 K=1,3
      DO 7 I=1,9
      JM=JR=0
      IF(GR(K,I,1).GT.GR(K,I,2)) IFLAG=1
      IF(GR(K,I,1).LT.GR(K,I,2)) IFLAG=0
```

```
      DO 6 I1=2,N
      IF(IFLAG.NE.1) GO TO 9
      IF(GR(K,I,(I1-1)).GE.GR(K,I,I1)) GO TO 6
      JM=JM+1
      XMIN(K,I,JM)=GR(K,I,(I1-1))
      IFLAG=0
      GO TO 6
9     IF(GR(K,I,(I1-1)).LE.GR(K,I,I1)) GO TO 6
      JR=JR+1
      XMAX(K,I,JR)=GR(K,I,(I1-1))
      IFLAG=1
6     CONTINUE
      IC1(K,I)=JM
      IC2(K,I)=JR
7     CONTINUE
      PRINT 155
155   FORMAT(//,5X,"CONSTANT GAIN VALUES"/,5X,"AVERAGE VALUE",5X,"NO OF
     +MAX + MIN",5X, "TOTAL OF MAX AND MIN",/)
      DO 16 K=1,3
      KR=2*K
      IF(KR.EQ.6) KR=7
      PRINT*,"            K(",KR,",X)"
      DO 16 I=1,9
      XTEMP=0
      N=IC2(K,I)
      DO 17 I1=1,N
17    XTEMP=XTEMP+XMAX(K,I,I1)
      N1=IC1(K,I)
      DO 131 I1=1,N1
131   XTEMP=XTEMP+XMIN(K,I,I1)
      N=N+N1
      XTEM=XTEMP/N
      WRITE (7,135) XTEM
135   FORMAT(1PE13.4)
16    PRINT 132,XTEM,N,XTEMP
```

```
132   FORMAT(5X,1PE13.4,8X,I4,8X,1PE13.4)
      STOP "GRATEFULLY"
      END
```

Appendix C

FORTRAN Program for Performance Computation

The program listed is used for both constant gain and varying gain inputs for performance computations. The modification listed must be made for the program to work properly. For fixed gain calculations, only one set of data is read in; for varying gain calculations the entire gain schedule must be read into the program.

Both the input data and the forms of output are listed in the program.

```
      PROGRAM COST2(INPUT,OUTPUT,TAPE5,TAPE6,TAPE7)
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                      C
C     THIS PROGRAM COMPUTES THE PERFORMANCE OF THE SATELLITE FOR       C
C     EITHER A FIXED GAIN OR VARYING GAIN CONTROLLER.                  C
C     FOR VARYING GAIN COMPUTATIONS TAPE5 MUST BE PRODUCED BY THE      C
C     PROGRAM "GAIN".                                                  C
C     FOR FIXED GAIN COMPUTATIONS TAPE7 MUST BE PRODUCED BY PROGRAM    C
C     INVERT.                                                          C
C     INPUT DATA:                                                      C
C          E - ECCENTRICITY OF THE LUNAR ORBIT                         C
C          E1- ECCENTRICITY OF THE BARYCENTER ORBIT                    C
C          BETA- INCLINATION OF THE ECLIPTIC                           C
C          GAM - INITIAL LUNAR POSITION RELATIVE TO THE ROTATING FRAME C
C          ALF - INITIAL SUN DIRECTION RELATIVE TO INERTIAL FRAME      C
C          OM - RADIAN FREQUENCY DIFFERENCE                            C
C          X - INITIAL CONDITIONS ON STATE VARIABLES.                  C
C             **NOTE: X1,X3,X6 ARE POSITION STATES                     C
C                     X2,X4,X7 ARE VELOCITY STATES                     C
C          TF - FINAL TIME                                             C
C          T - INITIAL TIME                                            C
C          Q - WEIGHTING ON POSITION STATES                            C
C          ITYPE- INTEGER SPECIFYING TYPE COMPUTATION:                 C
C            0 - VARYING GAIN COMPUTATION                              C
C            1 - CONSTANT GAIN COMPUTATION                             C
C     OUTPUT :                                                         C
C          1.   PRINTOUT OF POSITION AND VELOCITY STATES,TOTAL VELOCITY C
C               INCREMENT, AND DRIFT FROM L4.                          C
C          2.   TAPE6 - CAN BE USED AS DATA FOR PLOTTING POSITION      C
C     AND/OR VELOCITY DATA.                                            C
C     SUBROUTINES USED:                                                C
C          1.   ODE (CC6600 LIBRARY) - INTEGRATES THE STATE VARIABLES  C
C          2.   FX - COMPUTES THE VALUE OF THE STATES AT EACH TIME     C
C               INTERVAL, DT, FOR INPUT TO ODE.                        C
C          3.   CONST - COMPUTES THE VALUES OF THE COEFFICIENTS OF THE C
```

```
C                "A" MATRIX AT EACH TIME, DT.                                C
C            4.  EANOM - COMPUTES THE ECCENTRIC ANOMALY GIVEN MEAN           C
C                MOTION AND ECCENTRICITY.                                    C
C                                                                            C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      COMMON/COM2/A(9,9),GG(9,9),AT,OM,ALF,GAM
      COMMON/COM3/F,E1,BETA,ALFD,GAMD
      DIMENSION Z(45),X(9),XX(9),G3(3,9)
      DIMENSION WK(1500),IWK(5)
      DIMENSION TR(11)
      DIMENSION C(21)
      DOUBLE PRECISION DV
      EXTERNAL FX,EANOM,CONST
      NAMELIST/N2/E,E1,BETA,GAM,ALF,OM,X,TF,T,Q,ITYPE
 010  FORMAT(I3)
      READ N2
      PRINT*,"      INPUT DATA:     E=  ",E,"      E1=   ",E1,"   GAM=   ",G
     +AM,"     ALF=  ",ALF,"    TF=  ",TF,"    BETA=  "
     +,BETA,"  Q=  ",Q
      PRINT*,"                        X(1)= ",X(1),"   X(2)= ",X(2),"   X(3)
     += ",X(3),"  X(4)= ",X(4)," X(6)= ",X(6),"  X(7)= ",X(7)
      PRINT 015
 015  FORMAT(//,50X,"***SATELLITE DRIFT PARAMETERS***")
      IF(ITYPE.EQ.1) PRINT 020
 020  FORMAT(65X,"FOR")
      IF(ITYPE.EQ.1) PRINT 019
 019  FORMAT(55X,"CONSTANT GAIN ELEMENTS",//)
      PRINT 014
 014  FORMAT(9X,4HTIME,4X,10HX-POSITION,3X,10HX-VELOCITY,3X,10HY-POSITIO
     +N,3X,10HY-VELOCITY,3X,10HZ-POSITION,3X,10HZ-VELOCITY,4X,8HSIN(OM1)
     +,4X,9H DELTA V ,6X,5HDRIFT,//)
      IFLAG=1
      RE=AE=1.E-7
      DO 1 J=1,9
 1    XX(J)=X(J)
```

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```
      DT=.1
      NF=(TF-T)/ABS(DT)+1.1
      IF(ITYPE.EQ.1) READ*, ((G3(K,J),J=1,9),K=1,3)
011   FORMAT(1P9E13.4)
      ALF=ALF*3.1415927/180.
      GAM=GAM*3.1415927/180.
      ALFD=ALF
      GAMD=GAM
      DO 3 I=1,9
      DO 3 J=1,9
3     A(I,J)=0.
      DV=0.$EPO=0.$TOUT=T+DT
      K=0.
      WRITE (6,010) NF
      IF(ITYPE.EQ.0) GO TO 400
      K=0
      UTOT=0
      DO 4 LA=1,3
      K=K+2
      IF(K.EQ.6) K=7
      DO 4 LB=1,9
4     GG(K,LB)=G3(LA,LB)
400   CONTINUE
      DO 40 MA=2,NF
      UX=UY=UZ=0.
      IF(ITYPE.EQ.1) GO TO 435
      READ (5,011) ((G3(K,J),J=1,9),K=1,3)
      K=0.
      DO 436 LA=1,3
      K=K+2
      IF(K.EQ.6) K=7
      DO 436 LB=1,9
436   GG(K,LB)=G3(LA,LB)
435   CONTINUE
      DO 41 LT=1,5,5
```

```
      CALL ODE(FX,11,TR,T,TOUT,RE,AE,IFLAG,WK,IWK)
      TOUT=T+DT
      ALF=ALFD-.0040216*T
      DO 405 L4=1,9
405   X(L4)=TR(L4)
      CTOT=TR(10)+TR(11)
021   FORMAT(/,5X,"TOTAL QUADRATIC COST:  ",1PE13.4,5X,"RSQ*DT: ",1PE13.
     +4,5X,"USQ*DT: ",1PE13.4,/)
      ANG=OM*T+ALF+GAM
      OM1=2.*ANG
      X(5)=1.
      X(8)=COS(OM1)
      X(9)=SIN(OM1)
      DO 5 L=1,9
      UX=UX-GG(2,L)*X(L)
      UY=UY-GG(4,L)*X(L)
5     UZ=UZ-GG(7,L)*X(L)
      USQ=UX**2+UY**2+UZ**2
      U=SQRT(USQ)
      DV=DV+U*DT
      RSQ=X(1)**2+X(3)**2+X(6)**2
      R=SQRT(RSQ)
      RP=AMAX1(R,RPO)
      RPO=RP
      WRITE (6,012) T,X(1),X(2),X(3),X(4),X(6),X(7),X(9),DV,R
012   FORMAT(0PF7.2,1P9E13.4)
      PRINT 013,NA,T,X(1),X(2),X(3),X(4),X(6),X(7),X(9),DV,R
      AAA=AMOD(T,5.)
      IF(AAA.LE.5..AND.AAA.GE.4.99999) PRINT 021,CTOT,TR(10),TR(11)
41    CONTINUE
      UTOT=UTOT+U
013   FORMAT(I7,0PF8.2,1P9E13.4)
40    CONTINUE
      AVG=DV/TF
      PRINT 191,AVG,DV
```

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```
191   FORMAT(//,5X,"AVERAGE THRUST:  ",E13.4,5X,"DELTA V:  ",E13.4)
      UTO=UTOT/(NF-1)
      PRINT 195,UTO,UTOT,NF
195   FORMAT(//,5X,"AVERAGE THRUST*  ",E13.4,/,5X,"TOTAL VELOCITY  ",
     +E13.4,"  SUBDIVISIONS  ",I4)
      PRINT*,"     MAX RADIUS FROM L4  ",RPO
      PRINT 121,CTOT,TR(10),TR(11)
      STOP  "GRACEFULLY"
      END
```

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                      C
C     ****** SUBROUTINE FX ******                                      C
C                                                                      C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      SUBROUTINE FX(T,X,DX)
      COMMON/COM2/A(9,9),GG(9,9),AT,OM,ALF,GAM
      COMMON/COM3/F,E1,BETA,ALFD,GAMD
      DIMENSION X(11),DX(11),AX(9,9),U(3)
      DIMENSION C(21)
      ALF=ALFD-.0040216*T
      ANG=OM*T+ALF+GAM
      OM1=2.*ANG
      CALL CONST(C,E,E1,BETA,T)
      A(1,2)=1.
      A(2,1)=C(2)+C(3)*COS(OM1)
      A(2,3)=C(4)-C(5)*SIN(OM1)
      A(2,4)=C(1)
      A(2,5)=C(6)
      A(2,6)=-C(15)*(1+COS(OM1))
      A(2,8)=C(7)
      A(2,9)=-C(8)
      A(3,4)=1.
      A(4,1)=C(9)-C(5)*SIN(OM1)
      A(4,2)=-C(1)
      A(4,3)=C(10)-C(11)*COS(OM1)
      A(4,5)=C(12)
      A(4,6)=C(19)*SIN(OM1)
      A(4,8)=-C(13)
      A(4,9)=-C(14)
      A(6,7)=1.
      A(7,1)=-C(15)*(1.+COS(OM1))
      A(7,3)=C(16)*SIN(OM1)
      A(7,5)=-C(17)
      A(7,6)=C(20)+C(21)*COS(OM1)
```

```
      A(7,8)=-C(17)
      A(7,9)=C(18)
      A(8,9)=-2.*OM
      A(9,8)=2.*OM
      X(5)=1.
      X(8)=COS(OM1)
      X(9)=SIN(OM1)
      DO 7 L1=1,3
    7 U(L1)=0.
      DO 3 L=1,9
    3 AX(1,L)=AX(3,L)=AX(5,L)=AX(6,L)=0.
      AX(1,2)=AX(3,4)=AX(6,7)=1.
      DO 4 L=1,9
      U(1)=U(1)-GG(2,L)*X(L)
      U(2)=U(2)-GG(4,L)*X(L)
      U(3)=U(3)-GG(7,L)*X(L)
      AX(2,L)=A(2,L)-GG(2,L)
      AX(4,L)=A(4,L)-GG(4,L)
    4 AX(7,L)=A(7,L)-GG(7,L)
      DO 5 L=1,7
      DX(L)=0.
      DO 5 K=1,9
    5 DX(L)=AX(L,K)*X(K)+DX(L)
      DX(10)=DX(1)**2+DX(3)**2+DX(6)**2
      DX(11)=U(1)**2+U(2)**2+U(3)**2
      RETURN
      END
```

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                        C
C      ****** SUBROUTINE CONST ******                                    C
C                                                                        C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      SUBROUTINE CONST(C,E,E1,BETA,T)
      COMMON/COM2/A(9,9),GG(9,9),AT,OM,ALF,GAM
      DIMENSION C(21)
      REAL NO
      OMEGA=.07480133
      ASUN=1.4959965E8
      AMASS=1/82.301
      AMSUN=328905.2
      AMOON=384400
      ASNORM=ASUN/(AMOON*(1-E))
      AM1=GAM*T
      CALL EANOM(AM1,E,UM)
      AM2=ALF+OMEGA*T
      CALL EANOM(AM2,E1,US)
      RS=ASNORM*(1-E1*COS(US))
      SUN=AMSUN/RS**3
      FM=1/(1-E*COS(UM))
      QM=1-2.*AMASS
      NO=SQRT(3.)*(1-E*COS(UM))/(2.*(1-E))
      E0=QM*NO/SQRT(3.)
      ESQ=SQRT(1-E**2)
      FE=1-E**2
      C(1)=2.*ESQ*FM**2
      C(2)=FE*FM**4-.25*FM**3-SUN*(1-3.*COS(2.*BETA))/4
      C(3)=.75*SUN*(1.+COS(2.*BETA))
      C(4)=1.29904*QM*FM**3-2.*E*ESQ*SIN(UM)*FM**4
      C(5)=1.5*SUN*COS(BETA)
      C(6)=SUN*(3.*COS(2.*BETA)-1)*E0/4
      C(7)=.75*SUN*(1.+COS(2.*BETA))*E0
      C(8)=1.5*SUN*COS(BETA)*NO
```

```
C(9)=1.29904*QM*FM**3+2.*E*ESQ*SIN(UM)*FM**4
C(10)=1.25*FM**3+EE*FM**4+0.5*SUN
C(11)=1.5*SUN
C(12)=.5*SUN*NO
C(13)=C(11)*NO
C(14)=C(5)*EO
C(15)=.75*SUN*SIN(2.*BETA)
C(16)=C(11)*SIN(BETA)
C(17)=C(15)*EO
C(18)=C(13)*SIN(BETA)
C(19)=.75*SUN*SIN(BETA)
C(20)=-FM**3-SUN*(1+3.*COS(2.*BETA))/4
C(21)=.75*SUN*(1-COS(2.*BETA))
RETURN
END
```

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                       C
C     ****** SUBROUTINE EANOM ******                                    C
C                                                                       C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      SUBROUTINE EANOM(AM,E,U)
      U0=AM+(E-(E**3)/8.+(E**5)/192.)*SIN(AM)+((E**2)/2.-(E**4)/6.+(E**6
     +)/48.)*SIN(2.*AM)+((3.*E**3)/8.-(27.*E**5)/128.)*SIN(3.*AM)+(
     +(E**4)/3.-(4.*E**6)/15.)*SIN(4.*AM)+(125.*E**5*SIN(5.*AM))/384.
  105 AM0=U0-E*SIN(U0)
      DELU=(AM-AM0)/(1-E*SIN(U0))
      IF(DELU.LT.1.E-8) GO TO 106
      U0=U0+DELU
      GO TO 105
  106 U=U0+DELU
      RETURN
      END
```

Vita

George DeFilippi, Jr. was born [redacted] PII Redacted

[redacted]. He graduated from [redacted] in 1965 and entered the Air Force Academy. In June of 1969 he graduated with a Bachelor of Science in Aeronautical Engineering and received his commission as a Second Lieutenant. He completed pilot training and received his wings in August 1970. He served a tour of duty in Southeast Asia as a Forward Air Controller. On completion of the remote tour he was assigned to Vance AFB as an Instructor Pilot. In June 1976 he entered the Air Force Institute of Technology School of Engineering.

He is married to [redacted] and they have [redacted]

This thesis has been typed by Mrs. Dolores W. Witt.

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In this work, the station keeping parameters at the earth-moon libration point, L4, were studied. First, the equations of motion for a three-dimensional, four body system with elliptical orbits were derived. These equations were then linearized about the L4 point; and optimal control theory was applied to obtain a linear feedback controller.

The major computations of the controller were associated with the gain matrix, which is the solution to the time varying Riccati equation. Because of the periodic nature of the time varying gains, it was felt that

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a modified (fixed gain) control could be used. The modified controller was found by computing the steady-state average of the time varying gains.

Several observations were made in studying the performance of the satellite in the vicinity of the L4 point. First, it was found that the modified controller was computationally much simpler than the optimum controller while providing near optimal performance. Second, there is approximately a linear relationship, up to a point, between station keeping cost and distance from the L4 point. Third, there are initial solar configurations which minimize station keeping costs.